TRANSACTIONS OF
THE
OPTICAL SOCIETY

VOL. XXX. 1928-29 No. 2.

THE MEASUREMENT OF AXIAL ABERRATIONS OF TELESCOPIC SYSTEMS OF SMALL MAGNIFICATION*

BY B. E. MOURASHKINSKY AND MISS M. SAVOSTIANOFF

[Communicated by Dr JOHN S. ANDERSON]

MS. received, 11th August, 1928. *Read and discussed, 13th December*, 1928.

ABSTRACT. The aim of this work is to give a method for the measurement of axial aberrations (spherical and chromatic) of a telescopic system of small magnification. The dioptric expression for an axial aberration is given and the special advantages of such a dioptric expression for the experimental measurement of aberrations are pointed out. The principal peculiarity of a dioptric expression for axial aberration is the fact that the shape of a curve of spherical aberration remains unchanged if the focussing is varied.

The arrangements used in measuring axial aberrations are described. The method of the measurement consisted in getting two out of focus photographs of a slit or pinhole placed in the focus of a collimating object-glass and illuminated by monochromatic light. The object-glass of a system to be measured was covered by a diaphragm with slit or round hole apertures. The distance between the two photographic plates and the distance between the traces for two definite slits or holes on both the plates having been measured, it was possible to get the value of the angle between a given ray and the axis of the system. By means of this angle and the exit height (in the exit pupil) we get the distance of the intersection of the exit ray with the axis of the system from the centre of the exit pupil. The values of these distances, as expressed in dioptres, for the series of rays for different entrance or exit heights and for different wave-lengths being known, the curves of spherical aberration can be traced.

The aberrations of a number of binocular telescopes were measured to test this method. The results of the measurements for three telescopes are given in the paper. The data for one of these telescopes are given in more detail to show the attainable accuracy of the measurements. The results obtained by measurements with a round hole and round apertures were compared with those obtained by means of a slit diaphragm. Also the results obtained in focussing the system 0 D (infinity) and $-\frac{1}{4}$ D were compared†. The discrepancies in both cases lie within the limits of error. The measurements of aberration

* First published (in Russian) in *Trans. Opt. Inst. Leningrad*, 4, No. 43 1928.

† [The curvature, expressed in dioptres, of the emergent wave corresponding to the central point of the field is adopted as a measure of the focussing adjustment.—ED.]

of one of the telescopes were compared with the aberration curve obtained by means of computations for the other telescope of the same binocular.

Two telescopes showed so-called axial astigmatism and axial coma. The presence of axial astigmatism and axial coma is caused by insufficiently careful adjustment and centering of the optical system of these telescopes.

Introduction

THE quality of a telescopic system is generally defined by the quality of an image given by such a system. The quality of an image is defined either by its general sharpness (qualitatively) or by the angular value of the least details distinguished by the eye when observing with such a system, i.e. by the resolving power (quantitatively). These two methods are generally used for testing and characterising the quality of a telescopic system*. We are not aware of any paper concerning the measurement of aberrations of a telescopic system as a whole other than that by Wetthauer, which deals principally with the astigmatism (for oblique bundles) of telescopic systems†. The present paper is an attempt to give a method of measuring the axial aberrations of telescopic systems.

Representation of Spherical Aberration

Axial aberrations (spherical and chromatic) of a telescopic system may be represented as follows. A telescopic system being absolutely free from spherical aberration, the bundle of monochromatic light entering the object-glass (entrance pupil) of the system parallel to its axis leaves the eyepiece (exit pupil) also as a parallel bundle (Fig. 1 *A*). It is assumed that the system is focussed for infinity, i.e. the back focus of the object-glass coincides with the front focus of the eyepiece. On the other hand, if spherical aberration is present, the same parallel bundle with the same focussing of the system will not leave the eyepiece as a parallel bundle. The rays that leave the eyepiece at different exit heights may have different inclinations to the axis of the system and consequently will intersect it at different points (Fig. 1 *B*).

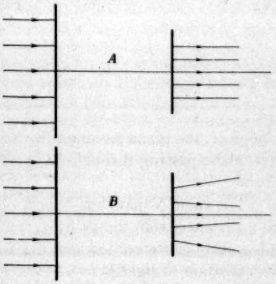

Fig. 1

The spherical aberration may be represented as a function of the height H of the entrance ray in the entrance pupil or the height $h = H/M$ of the exit ray in the exit pupil, where M is the magnification of the system for paraxial rays. The spherical aberration may be numerically expressed by the values of the angles formed

* Cf. Smith, "Apparatus for the Testing of Binocular Telescopes," *Journ. Opt. Soc. Amer.*, **2**, 3 1919 (76).

† Wetthauer, "Untersuchung des Bildes in Handfernrohren nach Methode der streifenden Abbildungen," *Z. f. Instrk.* **44** 1924 (189).

by the rays of different heights with the axis of the system. These values are generally given in arc minutes and seconds.

The spherical aberration may also be numerically expressed by means of the values of the distances from the point of intersection of the axis with the last surface of the eyepiece to the point of its intersection with a given ray entering the system at a given height. It is possible to take as reference plane not the plane tangent to the last vertex of the eyepiece, but the plane of the exit pupil. This, as will be shown below, is far more convenient for the method used. The above-mentioned distances are expressed either in line measures or in dioptres. By dioptre we understand the reciprocal of the distance of one metre.

The relation between the angular expression and the dioptric expression for the spherical aberration of a telescopic system may be represented in the form

$$N = \zeta \,.\, 1000/h \qquad \ldots\ldots(1),$$

where ζ is the angle (now in radians) of inclination of a given exit ray, h the height of this ray in millimetres, and N the number of dioptres. The value h may be taken in any plane chosen as reference plane; for instance, the back principal plane of the

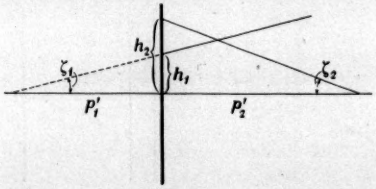

Fig. 2

eyepiece, the plane passing through the back vertex of the last lens of the eyepiece, the plane passing through the back focus of the eyepiece, or the plane of the exit pupil.

The value N may thus be referred to any of these planes taken as reference plane. If a distance from the point of intersection of the reference plane with the axis to the point of intersection of the axis with a given ray is read in the direction of the propagation of light it is considered positive and the value N corresponding to it will also be positive, and vice versa. In Fig. 2, N is negative for the ray crossing the reference plane at the height h_1 and positive for the ray h_2. It is assumed that the propagation of light is from left to right.

Plotting the entrance heights as ordinates and corresponding values of ζ or N as abscissae we get the curve of spherical aberration of a telescopic system. For the ideal (free from spherical aberration) telescopic system the angle ζ for all entrance heights will be zero, consequently N will also be zero and the curve of spherical aberration will be a straight line coinciding with the axis of ordinates.

If a telescopic system free from spherical aberration is not focussed for infinity, i.e. if the eyepiece is shifted from its initial position, a bundle of parallel rays that

enters the object-glass will give an image at a finite distance. The rays leaving the system at different heights will intersect the axis at one point though they will form different angles with it.

The position of the exit pupil and the position of the intersection point of a given ray with the axis for a telescopic system with or without spherical aberration may be expressed as follows. Let F and f be the focal lengths of the object-glass and the eyepiece of the system, Δ the linear shift of the eyepiece from the focussing for infinity to the focussing for a finite distance, and x_1' and x_p' the distances of the intersection point of a given ray with the axis and of the exit pupil from the back focus of the eyepiece. Then we have

$$x_1' = -f^2/\Delta, \quad x_p' = -f^2/(F + \Delta) \qquad \ldots\ldots(2).$$

The distance of the intersection point of a ray in question from the exit pupil will be

$$p' = x_1' - x_p' = -f^2 F/(\Delta^2 + \Delta F) \qquad \ldots\ldots(3).$$

Hence

$$N = 1000/p' = -(\Delta^2 + \Delta F) \cdot 1000/f^2 F \qquad \ldots\ldots(4).$$

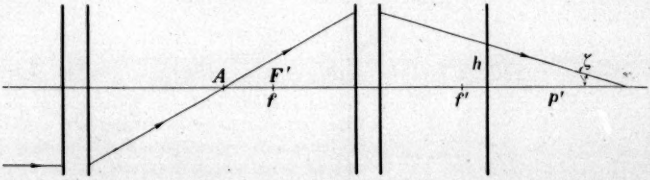

Fig. 3

In this case for a given value of Δ the value of N will be constant for all rays with different entrance heights. If the eyepiece is pushed in, Δ is positive and p' and N are negative. If the eyepiece is pulled out, Δ is negative and p' and N are positive. When $\Delta = 0$ (focussing for infinity) we get

$$x_1' = \infty, \quad x_p' = -f^2/F,$$
$$p' = \infty, \quad N = 0.$$

When spherical aberration is present we discuss the problem as follows. Let us assume that the ray emerging at the height h (the system is focussed for infinity) intersects the axis behind the eyepiece at the angle ζ (Fig. 3). The path of this ray through the system may then be represented as is shown in Fig. 3, i.e. the ray entering the system parallel to its axis intersects the axis before the focus (for paraxial rays) F' of the object-glass.

Putting $fA = x_1$, we get

$$x_1' = -f^2/x_1, \quad x_p' = -f^2/(F + x_1), \quad p' = -f^2 F/(x_1^2 + x_1 F),$$
$$N = -(x_1^2 + x_1 F) \cdot 1000/f^2 F \qquad \ldots\ldots(5).$$

If the eyepiece of the same system is shifted by the amount Δ, we get (see Fig. 4)

$$fA = x_1 + \Delta, \quad x_1' = f^2/(x_1 + \Delta), \quad x_2' = -f^2/(F + x_1 + \Delta),$$

$$p' = -f^2F/\{(x_1 + \Delta)^2 + (x_1 + \Delta)F\}, \quad N = -\{(x_1 + \Delta)^2 + (x_1 + \Delta)F\} \cdot 1000/f^2F \quad \ldots\ldots(6).$$

Suppose we have two rays entering the system at heights H_1 and H_2 (Fig. 4). Let $F'A_2 = x_1$ for the ray H_1 and $F'A_1 = x_2$ for the ray H_2. Let us also have two cases of focussing Δ_1 and Δ_2. From the preceding we can write

$$N_1' = -\{(x_1 + \Delta_1)^2 + (x_1 + \Delta_1)F\} \cdot 1000/f^2F \quad \ldots\ldots(7),$$

$$N_2' = -\{(x_2 + \Delta_1)^2 + (x_2 + \Delta_1)F\} \cdot 1000/f^2F \quad \ldots\ldots(8),$$

$$N_1'' = -\{(x_1 + \Delta_2)^2 + (x_1 + \Delta_2)F\} \cdot 1000/f^2F \quad \ldots\ldots(9),$$

$$N_2'' = -\{(x_2 + \Delta_2)^2 + (x_2 + \Delta_2)F\} \cdot 1000/f^2F \quad \ldots\ldots(10).$$

In these expressions N_1' and N_1'' are referred to the ray H_1 for the focussings of the system Δ_1 and Δ_2, and N_2' and N_2'' are referred to the ray H_2 for the same focussings. If, for instance, the ray H_2 has a maximum or a minimum (as compared

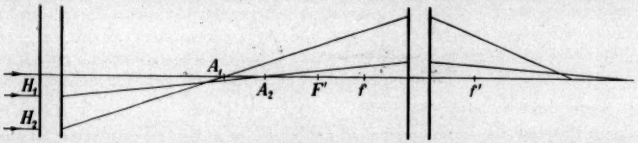

Fig. 4

with other rays emerging at different heights) value of the angle of inclination to the axis and the ray H_1 is a paraxial ray, then the expression $N_2' - N_1'$ gives the maximum value of spherical aberration for the focussing Δ_1. The expression $N_2'' - N_1''$ gives the analogous value for the focussing Δ_2. From equations (7) to (10) we have for these values

$$N_2' - N_1' = -\{(x_2^2 - x_1^2) + 2\Delta_1(x_2 - x_1) + (x_2 - x_1)F\} \cdot 1000/f^2F \quad \ldots\ldots(11),$$

$$N_2'' - N_1'' = -\{(x_2^2 - x_1^2) + 2\Delta_2(x_2 - x_1) + (x_2 - x_1)F\} \cdot 1000/f^2F \quad \ldots\ldots(12).$$

These two expressions show that the value of the spherical aberration of a telescopic system will not be altogether constant for the focussings Δ_1 and Δ_2. The difference between these values is given by

$$(N_2'' - N_1'') - (N_2' - N_1') = -2(\Delta_2 - \Delta_1)(x_2 - x_1) \cdot 1000/f^2F \quad \ldots\ldots(13).$$

This difference is generally very small and may be neglected in the majority of cases. As an example let us take the system $F = 157{\cdot}83$ mm., $f = 26{\cdot}24$ mm., and $\Delta_1 = 0$. In the first case the system is focussed for infinity for paraxial rays, consequently $x_1 = 0$. In the second case the system is focussed -4 D. For this case the expression (4) gives $\Delta_2 = 2{\cdot}71$ mm. Suppose that the spherical aberration of this system for $\Delta_1 = 0$ is equal to $\cdot 5$ D; then according to the formula (5) we have $x_2 = {\cdot}34_5$ mm. Introducing these values in (13) we find that the difference

between the values of the spherical aberration of this system for the focussing for infinity and the focussing -4 D will be ·017 D. If instead of the value ·5 D for the spherical aberration we suppose it equal to ·7 D, then $x_2 = \cdot 48$ mm. and the above-mentioned difference is ·024 D.

Thus we can say that *the form of the curve of spherical aberration of a telescopic system expressed in dioptric measure does not practically depend on the focussing of the system*. This, as will be shown below, presents special advantages for the experimental measurement of spherical aberration*.

For the system of the last example the value N_2' for some given ray and for the focussing $\Delta_1 = 0$ is equal to ·5 D. The value N_2'' for the same ray, but for the focussing -4 D, may be found from (10) and (5); it is given by

$$N_2'' = -\left[\frac{x_2^2 + x_2 F}{f^2 F} + \frac{\Delta_2^2 + \Delta_2 F}{f^2 F} + \frac{2x_2\Delta_2}{f^2 F}\right] . 1000 \qquad \text{.....(14).}$$

In this case the value ·017 D is just the value we may neglect, taking the plane of the exit pupil as a reference plane.

If we take as a reference plane not the plane of the exit pupil but that passing through the back focus of the eyepiece and refer to it the values p' and N, we get

$$N_1' = -(\Delta_1 + x_1)/f^2 \qquad \text{.....(15),}$$
$$N_2' = -(\Delta_1 + x_2)/f^2 \qquad \text{.....(16),}$$
$$N_1'' = -(\Delta_2 + x_1)/f^2 \qquad \text{.....(17),}$$
$$N_2'' = -(\Delta_2 + x_2)/f^2 \qquad \text{.....(18).}$$

From these we have $(N_2'' - N_1'') - (N_2' - N_1') = 0$ .....(19),

i.e. in this case the above-mentioned peculiarity of a dioptric expression for the spherical aberration of a telescopic system is fulfilled not only approximately but quite exactly. It is not convenient, however, to take the plane passing through the back focus of the eyepiece as a reference plane, because in this case we have to find its position experimentally and from that to get the heights of its intersection with the given rays emerging from the system.

The focussing and the value of spherical aberration being varied, not only the position of the exit pupil is varied but also the heights in the exit pupil. These variations are practically not so great and may in most cases be neglected. If H is the height of a ray in the entrance pupil, h_0 the height of this ray in the exit pupil of a telescopic system without spherical aberration, and M the magnification of the system for paraxial rays, then we have

$$h_{01} = -H_1/M \qquad \text{.....(20),}$$
$$h_{02} = -H_2/M \qquad \text{.....(21),}$$
$$h_1' = -H_1 f/(F + x_1 + \Delta_1) \qquad \text{.....(22),}$$
$$h_2' = -H_2 f/(F + x_2 + \Delta_1) \qquad \text{.....(23),}$$
$$h_1'' = -H_1 f/(F + x_1 + \Delta_2) \qquad \text{.....(24),}$$
$$h_2'' = -H_2 f/(F + x_2 + \Delta_2) \qquad \text{.....(25).}$$

* This peculiarity of dioptric expression for spherical aberration of a telescopic system was pointed out by A. Toudorovsky in his unpublished paper "On the Errors of the Optical System of a Binocular of 6 × Magnification," read at the 57th Meeting of the Russian Optical Society, 2nd July, 1924.

From (21), (23), and (25) we get

$$h_2'' = h_2' \left[\frac{F + x_2 + \Delta_1}{F + x_2 + \Delta_2}\right] = h_2' \left[1 - \frac{\Delta_2 - \Delta_1}{F + x_2 + \Delta_2}\right] \quad \ldots\ldots(26),$$

$$h_2'' = h_{02} \left[\frac{Mf}{Mf + x_2 + \Delta_2}\right] \quad \ldots\ldots(27).$$

From the data of the last example we have

$$h_2'' = \cdot 983\, h_2', \quad h_2'' = \cdot 981\, h_{02}.$$

The difference of 2 per cent. of height of the exit ray may be practically neglected.

One of the principal and immediate consequences of the above-mentioned peculiarity of a dioptric expression for the spherical aberration of a telescopic system is the possibility of shifting the axis of ordinates parallel to itself without changing the shape of the aberration curve. Thus it is possible to make the axis of ordinates pass through any point of the aberration curve, for instance through the point of the curve corresponding to zero height or to the height of the edge of the object-glass. In other words, having obtained the curve of spherical aberration for any arbitrary focussing of the system, we may immediately obtain the value of the spherical aberration of the same system as focussed for infinity for paraxial or edge rays. It is this fact that presents a great advantage in measuring spherical aberration, for it is very difficult, even if possible, experimentally to focus the system for infinity for paraxial or edge rays by any simple method.

Representation of Chromatic Aberration

The chromatic aberration of a telescopic system may be represented as follows. A ray of white light entering an ideal (achromatised) telescopic system parallel to its axis at the height H emerges from the system also parallel to the axis. In the case of a non-achromatised system the same ray emerges from the system as a number of rays forming different angles with the axis dependent on wave-lengths. The angle between two chosen exit rays, say for the lines C and F, may be taken as a measure of chromatic aberration. If, for instance, the inclination of the exit ray for the line C is ζ_1, and that for the line F is ζ_2, then $\alpha = \zeta_1 - \zeta_2$ will give the value of chromatic aberration for a given entrance height. The expression for chromatic aberration may be given in angular measure as well as in dioptres. The chromatic aberration may be represented by a curve, the wave-lengths being plotted as abscissae and the values of ζ (or corresponding dioptric values N) for every given entrance height as ordinates. The chromatic aberration may also be represented by the curves of spherical aberration for a number of wave-lengths.

Method of Measurement

The measurement of the spherical aberration of a telescopic system was carried out in the following manner. Let us assume that the telescopic system is focussed for infinity for paraxial rays. A bundle of rays of monochromatic light enters the object-glass of the system parallel to its axis. The emerging rays have different

inclinations to the axis (Fig. 5). We place a photographic plate behind the system at A_1, then another plate at A_2, and select two rays entering symmetrically with respect to the axis at the height H. Thus we have two points (spots) on each plate. Knowing the distance between the positions of the plates and measuring the distance between two spots on each photograph we can obtain the angle ζ. Putting the distance a_1a_2 equal to e_1 and the distance $a_1'a_2'$ equal to e_2 and denoting the distance between the plates by l, we can write

$$\zeta = (e_2 - e_1)/2l \qquad \text{.....(28).}$$

The values of ζ, the entrance height H, and the magnification M being known, we can get the value of p' (the distance of the intersection point of the exit ray with

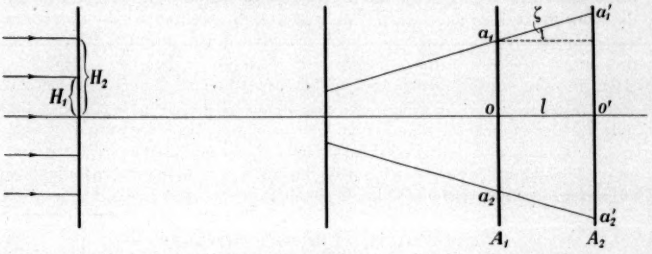

Fig. 5

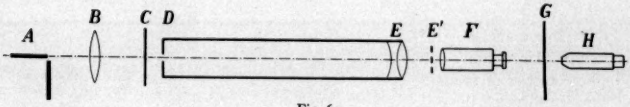

Fig. 6

the axis from the centre of the exit pupil) and from this we obtain the value of N. The latter value will be given by

$$N = (e_2 - e_1)\, M \,.\, 1000/2Hl \qquad \text{.....(29).}$$

In this, as well as in all the expressions below, all the linear measures are given in millimetres.

If we have the values of N for different entrance heights we can obtain the curve of spherical aberration of our telescopic system.

To test this method as to how far it is applicable, measurements of the axial aberrations of a number of binocular telescopes were made. The arrangement for these measurements was as follows (Fig. 6). An arc A (15–20 amp.) was focussed by the condenser B on the diaphragm D in the focal plane of the object-glass E of the collimator DE. The light filter C was placed between the condenser and the

diaphragm. The diaphragm was either a pinhole of diameter about ·06 mm. or a slit of breadth about ·09 mm. The following light filters were used:

	Limits of transparency	Mean value
	($m\mu$)	($m\mu$)
Blue *F*	450–525	487
Green 525	490–560	525
Yellow *D*	555–630	592
Red *C*	600–700	650

These light filters were prepared in the photographic laboratory of the Optical Institute, Leningrad. The object-glass of the collimator was a two-lens apochromat of C. Zeiss. Its focal length was measured by means of a large focometer of "Askania-Werke" and an Abbe focometer of C. Zeiss, the results being as follows:

Focometer "Askania-Werke"	I	1236·5
" "	II	1236·3
" "	III	1236·8
Abbe focometer		1236·6
	Mean...	1236·5 mm.

The spherical aberration of this object-glass, as measured by Miss M. Romanoff, does not exceed ·2 mm.

The collimator diaphragm has a rack and pinion focussing adjustment with a millimetre scale and vernier. Behind the collimator there was the support for the telescope *F* to be tested. This support had the necessary appliances for adjusting and centering the telescope. The plate-holder *G* was placed behind the telescope. The focussing microscope *H* was rigidly connected to the plateholder and was placed behind it. All these parts were placed on an optical bench.

The diaphragm *E'*, which was of the same type as that used in the testing of object-glasses by Hartmann's method with a number of round apertures, was placed on the object-glass of the telescope to be tested. The diameter of these apertures was 3 mm. In this case the collimator diaphragm was provided with a pinhole. In most measurements other diaphragms with a number of slits were used either with vertical slits (position 0°) the collimator slit being also vertical, or with horizontal slits (position 90°) the collimator slit being also horizontal.

The distance between two photographs ought to be taken as large as possible, but this is practically not feasible, as the resulting diffraction pattern becomes so unsharp and diffuse with the increasing distance between the plates that accurate measurements become impossible. The first, inner photograph was generally taken in the exit pupil, the construction of the telescope permitting the plate to be placed in the plane of the exit pupil. This is not obligatory, but is convenient. The plate-holder being shifted, the second photograph was taken at that maximum distance which gives some guarantee of accurate measurements. In order to choose this position, a "focussing of the plate-holder" (moving it along the bench) was made by means of the above-mentioned microscope. This focussing is to give the position of the second plate at the maximum distance from the first plate at which it is

possible to have a rather sharp, measurable diffraction image. The movements of the plate-holder were read on the millimetre scale (with vernier) of the bench.

The diaphragm with round apertures, although used, proved to be inconvenient as it was desirable to have a maximum number of apertures (i.e. a maximum number of values of entrance heights), and then the apertures had to be very small, thus greatly decreasing sharpness of the diffraction pattern.

If we know the distance of the centre of a given aperture or a given slit from the axis, we know the entrance height of a given ray, as the diaphragms were placed immediately at the object-glass of the telescope (in the entrance pupil). The magnification of the telescope being known, we can at once obtain the height of the ray in question in the exit pupil. This is just why the plane of the exit pupil has been taken as the reference plane for the values of p' and N.

On the photographs taken with the pinhole diaphragm we have the diffraction spots analogous to those on the photographs taken by Hartmann's method for the testing of object-glasses. On the photographs taken with the slit diaphragm we have bright and dark lines with additional diffraction fringes. The measurements of the distances e_1 and e_2 were made by means of a large measuring machine of "Askania-Werke" with a variable width double-line graticule.

Most of the measurements were made with the telescope to be tested approximately focussed for -4 D; thus the second plate could be moved a little farther. In order to show that this manner of measurement is admissible, i.e. to verify that the shape of the curve of spherical aberration, as expressed in dioptres, is independent of the focussing of the telescope, several telescopes were measured with the focussings -4 D and 0 D. For this latter the telescopes were focussed by means of a collimator and a supplementary telescope (C. Zeiss apparatus for testing binoculars).

Results

We give here the results of measurements of axial aberration of three telescopes: (1) one telescope of a prismatic binocular 6–42 of a well-known German firm, (2) one telescope of a prismatic binocular 6–30 of the same firm, (3) one telescope of a prismatic binocular 8–40, with a large field of view, of the same firm. The data for the first telescope are given in more detail to illustrate the attainable accuracy.

The telescope 6–42. Let the values of N for the focussing 0 D be denoted by N_0' and N_{90}' and those for the focussing -4 D by N_0'' and N_{90}''. The values $A' = N_{90}' - N_0'$ and $A'' = N_{90}'' - N_0''$ give the values of axial astigmatism. These values for this telescope lie within the limits of errors of measurement; thus the values N_0', N_{90}', N_0'', and N_{90}'' are not given separately, only the mean values

$$N' = (N_0' + N_{90}')/2 \text{ and } N'' = (N_0'' + N_{90}'')/2$$

being given in Tables I to IV. Instead of A' and A'' are given the values $A'/2$ and $A''/2$ which may characterise the accuracy of the measurements. In the second column H is the entrance height. In the seventh column of Tables I, II and IV, and the fifth column of Table III are given the values of $N'' - N'$, i.e. the

Table I. *Diaphragm with slits*

(*Blue filter*)

	H	N'	$A'/2$	N''	$A''/2$	$N''-N'$	N_1'	N	N_0
1	1·25	− ·01	− ·01	− 4·02	·00	− 4·01	− ·03	− ·02	− ·02
2	2·5	− ·02	− ·02	− 3·99$_5$	− ·01$_5$	− 3·97$_5$	− ·00$_5$	− ·01	− ·01
3	3·75	+ ·01	·00	− 3·97	·00	− 3·98	+ ·02	+ ·01$_5$	+ ·01$_5$
4	5·0	+ ·03	− ·01	− 3·95	·00	− 3·98	+ ·04	+ ·03$_5$	+ ·03$_5$
5	6·25	+ ·05	·00	− 3·93	·00	− 3·98	+ ·07	+ ·06	+ ·06
6	7·5	+ ·08$_5$	− ·00$_5$	− 3·89$_5$	+ ·00$_5$	− 3·98	+ ·09$_5$	+ ·09	+ ·09
7	8·75	+ ·10$_5$	− ·00$_5$	− 3·87	·00	− 3·97$_5$	+ ·12	+ ·11	+ ·11$_5$
8	10·0	+ ·14	− ·01	− 3·85$_5$	− ·00$_5$	− 3·99$_5$	+ ·13$_5$	+ ·14	+ ·13$_5$
9	11·25	+ ·14	− ·02	− 3·84	·00	− 3·98	+ ·15	+ ·14$_5$	+ ·16
10	12·5	+ ·17$_5$	− ·00$_5$	− 3·81	+ ·01	− 3·98$_5$	+ ·18	+ ·18	+ ·18
11	13·75	+ ·18$_5$	·00	− 3·81	·00	− 3·99$_5$	+ ·19	+ ·19	+ ·19
12	15·0	+ ·20$_5$	·00	− 3·82$_5$	+ ·00$_5$	− 4·03	+ ·16$_5$	+ ·18$_5$	+ ·20$_5$
13	16·25	+ ·19	+ ·02	− 3·82	—	− 4·01	+ ·17	+ ·18	+ ·19$_5$
14	17·5	+ ·19	·00	− 3·81	—	− 4·00	+ ·18	+ ·18$_5$	+ ·18$_5$
15	18·75	+ ·19	− ·01	− 3·83	—	− 4·02	+ ·16	+ ·17$_5$	+ ·16$_5$

Mean − 3·99 ± ·01

Distance between the plates in mm.

	0°	90°
0 D	48·3	41·0
− 4 D	59·1	59·9

Table II. *Diaphragm with slits*

(*Green filter*)

	H	N'	$A'/2$	N''	$A''/2$	$N''-N'$	N_1'	N	N_0
1	1·25	− ·02$_5$	+ ·00$_5$	− 4·01	·00	− 3·98$_5$	− ·02	− ·02	− ·02
2	2·5	·00	·00	− 3·99	+ ·01	− 3·99	·00	·00	·00
3	3·75	·00	+ ·01	− 3·99	+ ·02	− 3·99	·00	·00	+ ·03
4	5·0	+ ·06	·00	− 3·93	− ·02	− 3·99	+ ·06	+ ·06	+ ·06
5	6·25	+ ·07	·00	− 3·90$_5$	− ·00$_5$	− 3·97$_5$	+ ·08$_5$	+ ·08	+ ·09
6	7·5	+ ·12	·00	− 3·86	·00	− 3·98	+ ·13	+ ·12$_5$	+ ·12$_5$
7	8·75	+ ·16	·00	− 3·84$_5$	+ ·00$_5$	− 4·00$_5$	+ ·14$_5$	+ ·15	+ ·15$_5$
8	10·0	+ ·19	·00	− 3·79$_5$	+ ·01$_5$	− 3·98$_5$	+ ·19$_5$	+ ·19	+ ·19
9	11·25	+ ·23$_5$	+ ·00$_5$	− 3·79	+ ·01	− 4·02$_5$	+ ·20	+ ·22	+ ·22$_5$
10	12·5	+ ·25	·00	− 3·74	+ ·02$_5$	− 3·99$_5$	+ ·24$_5$	+ ·25	+ ·25
11	13·75	+ ·28	·00	− 3·73	− ·02	− 4·01	+ ·26	+ ·27	+ ·27
12	15·0	+ ·29$_5$	+ ·00$_5$	− 3·70$_5$	·00	− 4·00	+ ·28$_5$	+ ·29	+ ·29
13	16·25	+ ·30	+ ·01	− 3·69$_5$	—	− 3·99$_5$	+ ·29$_5$	+ ·30	+ ·30$_5$
14	17·5	+ ·31	·00	− 3·68	—	− 3·99	+ ·31	+ ·31	+ ·31$_5$
15	18·75	+ ·33	·00	− 3·66	—	− 3·99	+ ·33	+ ·33	+ ·31$_5$

Mean − 3·99 ± ·01

Distance between the plates in mm.

	0°	90°
0 D	42·2	43·2
− 4 D	52·8	53·2

Table III. *Diaphragm with slits*

(*Yellow filter*)

	H	N'	N''	$N''-N'$	N_1'	N	N_0
1	1·25	− ·15$_5$	− 4·11?	− 3·95$_5$	− ·12	− ·14	− ·15$_5$
2	3·75	− ·10$_5$	− 4·11	− 4·00$_5$	− ·12	− ·11	− ·11
3	6·25	− ·04	− 4·04	− 4·00	− ·05	− ·04$_5$	− ·04$_5$
4	8·75	+ ·03$_5$	− 3·96	− 3·99$_5$	+ ·03	+ ·03	+ ·03
5	11·25	+ ·12	− 3·88	− 4·00	+ ·11	+ ·11$_5$	+ ·11
6	13·75	+ ·16	− 3·82	− 3·98	+ ·17	+ ·16$_5$	+ ·17
7	16·25	+ ·21	− 3·78	− 3·99	+ ·21	+ ·21	+ ·21$_5$
8	18·75	+ ·22	− 3·79	− 4·00	+ ·20	+ ·21	+ ·24$_5$

Mean − 3·99 ± ·01

Distance between the plates in mm.

0 D	65·2
− 4 D	61·8

Table IV. *Diaphragm with slits*

(*Red filter*)

	H	N'	$A'/2$	N''	$A''/2$	$N''-N'$	N_1'	N	N_0
1	1·25	− ·30$_5$	+ ·00$_5$	− 4·33	− ·02	− 4·02$_5$	− ·33	− ·32	− ·31$_5$
2	2·5	− ·29	·00	− 4·29$_5$	− ·00$_5$	− 4·00$_5$	− ·29$_5$	− ·29	− ·29$_5$
3	3·75	− ·27	·00	− 4·26$_5$	+ ·00$_5$	− 3·99$_5$	− ·26$_5$	− ·27	− ·27
4	5·0	− ·22$_5$	+ ·01$_5$	− 4·24$_5$	+ ·00$_5$	− 4·02	− ·24$_5$	− ·23$_5$	− ·23$_5$
5	6·25	− ·20$_5$	− ·00$_5$	− 4·20	− ·01	− 4·00$_5$	− ·20	− ·20	− ·20
6	7·5	− ·15$_5$	+ ·00$_5$	− 4·14$_5$	+ ·01$_5$	− 3·99	− ·14$_5$	− ·15	− ·15$_5$
7	8·75	− ·12	·00	− 4·10$_5$	− ·00$_5$	− 3·98$_5$	− ·10$_5$	− ·11	− ·11
8	10·0	− ·07	·00	− 4·06$_5$	+ ·00$_5$	− 3·99$_5$	− ·06$_5$	− ·07	− ·07
9	11·25	− ·01$_5$	+ ·00$_5$	− 4·01	·00	− 4·00$_5$	− ·01	− ·01	− ·01
10	12·5	+ ·05	+ ·01	− 3·95	+ ·01	− 4·00	+ ·05	+ ·05	+ ·05
11	13·75	+ ·09$_5$	− ·00$_5$	− 3·90	·00	− 3·99$_5$	+ ·10	+ ·10	+ ·09$_5$
12	15·0	+ ·14	·00	− 3·84$_5$	+ ·01$_5$	− 3·98$_5$	+ ·15$_5$	+ ·15	+ ·15
13	16·25	+ ·17	·00	− 3·82$_5$	− ·00$_5$	− 3·99$_5$	+ ·17$_5$	+ ·17	+ ·17$_5$
14	17·5	+ ·20$_5$	− ·00$_5$	− 3·80	—	− 4·00$_5$	+ ·20	+ ·20	+ ·20
15	18·75	+ ·21	·00	− 3·79$_5$	—	− 4·00$_5$	+ ·20$_5$	+ ·21	+ ·22

Mean − 4·00 ± ·01

Distance between the plates in mm.

	0°	90°
0 D	35·7	35·2
− 4 D	41·9	42·2

differences between the values of N for the focussings -4 D and 0 D. The accuracy of measurement being sufficient and the statement of the constancy of the shape of the curve of spherical aberration being correct, the values of $N'' - N'$ must be constant for all the entrance heights and for all the wave-lengths. As is seen from the tables these values remain practically constant, thus showing the possibility of using a divergent bundle up to -4 D.

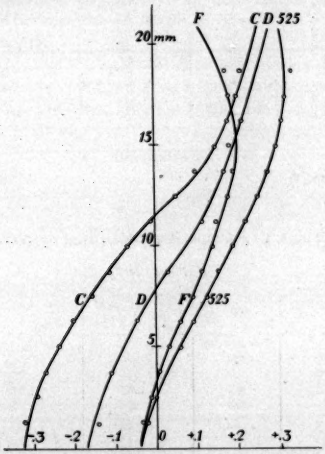

Fig. 7

We see from Tables I to IV that the difference $N'' - N'$ is practically constant, being equal to $-3{\cdot}99$ D. If we subtract this mean value from the values of N'' we get the value of N_1', the reduced value of N for the focussing 0 D. The mean of N' and N_1' gives the final value of N. Plotting the values of H as ordinates and the corresponding values of N as abscissae we get the curves of spherical aberration for the wave-length chosen. These curves are given in Fig. 7. The last column of Tables I to IV contains the values of N_0 as obtained from the curve. The values of N and N_0 differ only for some points, but even then the difference between them does not exceed $\cdot 01_5$ to $\cdot 02$ D.

All the above-mentioned photographs were taken with a slit diaphragm. In Tables V to VII the results of measurements by means of the diaphragm with round apertures are also given. The photograph with the round aperture diaphragm for the red filter was taken with the same focussing of the telescope as when photographing with the slit diaphragm. After this photograph had been taken the focussing of the telescope was occasionally, though very slightly, disarranged. In the

Table V. *Diaphragm with round apertures*

(*Blue filter*)

	H	ϕ	N	N_0	$N_0 - N$
		(degrees)			
1	2·9	90	+ ·09	− ·00$_5$	− ·09$_8$
2	5·0	0	+ ·15	+ ·03$_5$	− ·11$_5$
3	8·4	45	+ ·24	+ ·11	− ·13
4	9·5	90	+ ·26	+ ·13	− ·13
5	11·5	0	+ ·28	+ ·16$_5$	− ·11$_5$
6	15·0	45	+ ·34	+ ·20$_5$	− ·13$_5$
7	16·2	90	+ ·32	+ ·19$_5$	− ·12$_5$
8	18·2	0	+ ·30	+ ·17	− ·13

Mean − ·12 ± ·01

l = 63·9 mm.

Table VI. *Diaphragm with round apertures*

(*Green filter*)

	H	ϕ	N	N_0	$N_0 - N$
		(degrees)			
1	2·9	90	·00 ?	+ ·00$_5$	—
2	5·0	0	+ ·22	+ ·06	− ·16
3	8·4	45	+ ·25	+ ·14$_5$	− ·10$_5$
4	9·5	90	+ ·32	+ ·17$_5$	− ·14$_5$
5	11·5	0	+ ·32	+ ·23	− ·09
6	15·0	45	+ ·41	+ ·29	− ·12
7	16·2	90	+ ·43$_5$	+ ·30$_5$	− ·13
8	18·2	0	+ ·40	+ ·31$_5$	− ·08$_5$

Mean − ·12 ± ·02

l = 62·2 mm.

Table VII. *Diaphragm with round apertures*

(*Red filter*)

	H	ϕ	N	N_0	$N_0 - N$
		(degrees)			
1	5·0	0	− ·23	− ·23$_5$	− ·00$_5$
2	9·5	90	− ·08	− ·08$_5$	− ·00$_5$
3	11·5	0	+ ·01	·00	− ·01
4	15·0	45	+ ·15	+ ·15	·00
5	18·2	0	+ ·21	+ ·21$_5$	+ ·00$_5$

Mean ·00

l = 60·5 mm.

last column the differences $N_0 - N$ are given, N_0 being the value of N as obtained from the measurements with the slit diaphragm and N being that for the round aperture diaphragm. As was to be expected, the difference $N_0 - N$ for the red filter is equal to zero. For the blue and green filters the difference $N_0 - N$ is not equal to zero but remains constant for these two filters and equal to − ·12 D. This latter value just gives the difference of focussings for the red and two other light filters, i.e. the value of an accidental displacement of the eyepiece. All these data show a very good coincidence between the results obtained with the slit diaphragm and the round aperture diaphragm.

From Fig. 7 we have that the spherical aberration of this telescope, i.e. the difference between the value of N for paraxial rays and its maximum value, is: For the blue filter, + ·24 D; for the green filter, + ·34 D; for the yellow filter, + ·44 D; and for the red filter, + ·57 D.

Table VIII

(*Blue filter*)

H	N_0	N_e	$N_e - N_0$
0	− ·03	− ·34	− ·31
5	+ ·03$_5$	− ·30	− ·33$_5$
10	+ ·13$_5$	− ·19$_5$	− ·33
12·5	+ ·18	− ·12$_5$	− ·30$_5$
15	+ ·20$_5$	− ·10	− ·30$_5$
20	+ ·13$_5$	− ·17$_5$	− ·31

Mean − ·32 ± ·02

(*Green filter*)

H	N_0	N_e	$N_e - N_0$
0	− ·03	− ·38	− ·35
5	+ ·06	− ·32$_5$	− ·38$_5$
10	+ ·19	− ·18$_5$	− ·37$_5$
12·5	+ ·25	− ·11	− ·36
15	+ ·29	− ·04	− ·33
20	+ ·31	− ·02$_5$	− ·33$_5$

Mean − ·35$_5$ ± ·02

(*Yellow filter*)

H	N_0	N_e	$N_e - N_0$
0	− ·16$_5$	− ·50	− ·33$_5$
5	− ·08	− ·44	− ·36
10	+ ·07	− ·28	− ·35
12·5	+ ·14$_5$	− ·18	− ·32$_5$
15	+ ·19	− ·08$_5$	− ·27$_5$
20	+ ·27	− ·01	− ·28

Mean − ·32 ± ·03

(*Red filter*)

H	N_0	N_e	$N_e - N_0$
0	− ·32$_5$	− ·66	− ·33$_5$
5	− ·23$_5$	− ·59$_5$	− ·36
10	− ·07	− ·41	− ·34
12·5	+ ·05	− ·30	− ·35
15	+ ·15	− ·18$_5$	− ·33$_5$
20	(+ ·24$_5$)	− ·02$_5$	(− ·27)

Mean − ·34 ± ·01

The results of these measurements were compared with the data obtained by computation for the second telescope of the same binocular. These computations were made at the Computing Bureau of the Optical Institute, Leningrad, all the necessary data of this system (radii of curvature, refraction indices, and thicknesses) being known from previous measurements. It is hardly possible to expect two telescopes of the same binocular, even of a first-class firm, to have quite the same values of aberration, but the comparison of the value of aberration of one telescope obtained experimentally with that of the other telescope obtained by computation may give some characteristics of this method of measurement. The computed values of spherical aberration have been obtained for the lines F, 530$m\mu$, D, and C, the wave-lengths of which are close to the mean values of spectral transparency of our

light filters. This comparison is given in Table VIII, where N_0 is the observed value of N and N_e the computed value; the values of N_0 for 0 and 20 mm. have been obtained by extrapolation. As is seen from this table, while the mean values of $N_e - N_0$ are rather close to one another, the individual differences show some discrepancies.

The curves of chromatic aberration have been obtained directly. These curves are given in Fig. 8 for the entrance heights 5, 10, 15, and 20 mm. The curves show less achromatisation with decreasing entrance height, i.e. the achromatisation (as expressed in dioptres) is fulfilled to a greater extent for edge rays than for central rays.

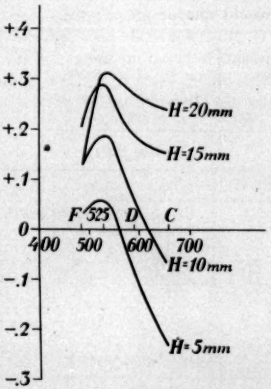

Fig. 8

The telescope 6–30. The photographs for this telescope were made with the slit diaphragms and with the same light filters; the focussing was about − 4 D. All the data of the measurements are not given here. The curves of spherical aberration of this telescope are given only for two light filters (Fig. 9), corresponding to the lines *F* and *C*, as the figure would be too complicated if the curves for all the light filters were drawn (see below). These curves show two peculiarities. First it is to be seen that the telescope has astigmatism for the axial bundle. The curves for the horizontal diameter are given as continuous lines and for the vertical diameter as dotted lines. As may be seen from the curves the axial astigmatism attains the value ·1 D. Besides that the curves for the horizontal diameter become double with increasing separation as the entrance height increases. This may be explained by the fact that two rays entering at the same height *H* but at different sides of the axis (Fig. 5) after having left the system intersect the axis in two different points, i.e. the distance $o'a_1'$ is not equal to the distance $o'a_2'$ and their difference increases with the entrance height. This phenomenon of the exit rays propagating non-symmetrically with relation to the axis of the system may be called axial coma. As is seen from the curves (Fig. 9) the axial coma of this telescope reaches ·3 D for the red filter. We must mention here that for this telescope the axial astigmatism and the axial coma (for the horizontal diameter) have been found with other light filters, the curves for which are not given in Fig. 9.

The telescope 8–40. The photographs for this telescope were taken with the slit diaphragm as well as with the round aperture diaphragm, with the same light filters, the focussing being about − 4 D.

Fig. 10 gives the curve of spherical aberration for the green filter. This telescope has also axial astigmatism and axial coma. The axial coma was found both for the vertical and horizontal diameters. The axial astigmatism and the axial coma were

found also with other light filters. The axial astigmatism is comparatively not so large, but the axial coma reaches ·3 to ·4 D. The aberrations of the second telescope of the same binocular were known by computation, but the presence of axial astigmatism and axial coma of the telescope in question made the comparison of observed and computed values of aberration impossible.

The axial astigmatism and the axial coma are caused by unsatisfactory centering of the elements of the optical system of the telescope. The axial astigmatism of ·15 to ·20 D may be unnoticed in direct observation with such a telescope, but the axial coma of ·3 to ·4 D makes itself felt.

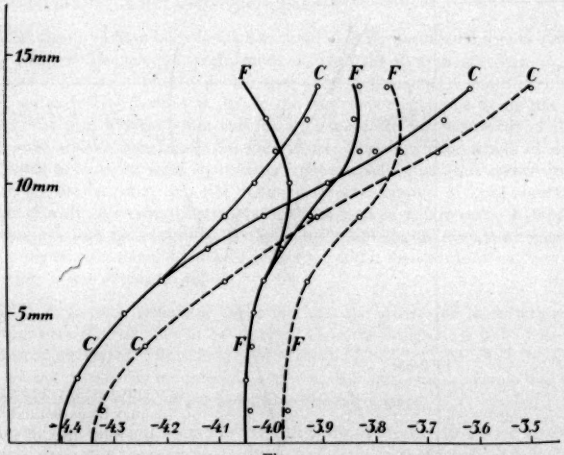

Fig. 9

The following experiment was made for direct observation of the axial coma. The two last telescopes were in turn placed in the instrument for testing the general definition and the resolving power of binocular telescopes. The test-chart was placed in the focus of the long focus collimator. This chart was observed through the telescope to be tested by means of a supplementary telescope. The object-glass of the telescope to be tested was then covered with an opaque diaphragm leaving only a small segment (about $\frac{1}{4}$ to $\frac{1}{3}$ of the radius of the object-glass) uncovered. This diaphragm was rotated by hand till the sharpest possible image of the test-chart was seen. Then the final focussing was made. The diaphragm was then rotated through 180°. In this case the image of the test-chart was unquestionably less sharp than for its previous position.

As is to be seen from all the above-mentioned results the spherical aberration

(as expressed in dioptric measure) of these telescopes is positive. The quality of a telescopic system (for axial bundles) is finally defined not by the values of spherical or chromatic aberration expressed in dioptric measure but by the angle of the diameter of the circle of least diffusion in the image space. The relation between this last angle and the dioptric value of aberration may be written

$$2\zeta = 2\,[N_m - N]\,h/1000,$$

where $N_m - N$ is the maximum value of spherical aberration. This formula shows that the same value of spherical aberration expressed in dioptric measure gives a

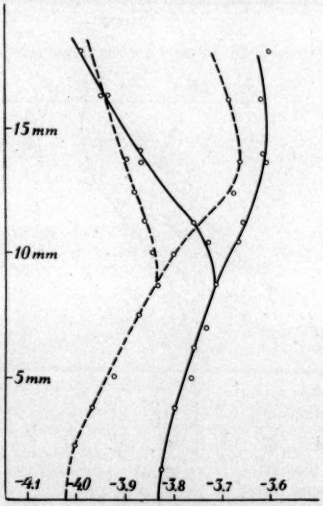

Fig. 10

larger angle of the circle of least diffusion for edge rays than for paraxial rays. Therefore it is preferable to focus a telescopic system having spherical aberration so that the value of N should be zero not for paraxial but for edge rays; in this case we should have a smaller circle of least diffusion for the same value of spherical aberration. Let us take, for instance, the first of the above-mentioned telescopes. The spherical aberration for the line D is equal to $+ \cdot 44$ D. The system is assumed firstly to be focussed so that N is equal to zero for paraxial rays, i.e. the axis of ordinates passes through the division (abscissa) $- \cdot 165$ D. The diameter of the exit pupil of this telescope is equal to 7 mm. From the last expression we have

$$2\zeta = 2 \times 3{\cdot}5 \times \cdot 44/1000 = \cdot 0031,$$

i.e. about 11 minutes. If this telescope is focussed so that N is equal to zero for

edge rays, the spherical aberration for paraxial rays is equal to $-\cdot 44$ D and the angular value of the circle of least diffusion is equal to only 2·3 minutes.

It is to be pointed out that the quality of the image given by a telescopic system of small magnification does not only depend on the perfection of the curves of spherical or chromatic aberration obtained by means of trigonometrical computation, but still more on the quite accurate execution of the requirements of computation and especially on accurate adjusting and centering.

This work was carried out at the Optical Institute, Leningrad.

DISCUSSION

Mr T. Smith: I have been much interested in the experimental results obtained by the authors. The general resemblance between the aberration curves for these telescopes and those of photographic lenses and other systems of finite focal length will be noticed. The theoretical part of the work however seems to me less satisfactory. The theory given is not that for a telescope but for an objective followed by a system entirely free from axial aberrations. The short focal length of an eyepiece tends to make its axial aberrations comparatively small, but against this we have to set the fact that it is not designed for the removal of axial aberrations, while in the objective special attention is given to this requirement. It may thus easily happen that the central aberrations attributable to the eyepiece are as great as, or even exceed, those due to the objective, and a theory which neglects them completely is not of general value.

Mr P. F. Everitt: Referring to the fact that the errors due to centering, etc., are larger than the residuals to be expected from the design, it is to be noted that the telescopes referred to appear to be of prismatic binocular type where the various parts are not necessarily mounted on a true optical axis, and the errors due to the mechanical imperfections of the mount may be fairly large.

Mr Mourashkinsky (communicated): In reply to Mr Everitt I have to say that all optical designs of the telescopic systems (prismatic systems of all kinds included) must be, and always are, based on the assumption that the system is *centred*. The experimental measurement of the axial aberrations of such a system shows the whole resulting effect caused by some imperfections in the optical design as well as by the defects of the execution of the optical and mechanical parts of the instrument.

I agree with Mr Smith's remarks that the central aberrations attributable to the eyepiece may be as great as, or even exceed, those due to the object-glass; moreover in the design of a telescopic system it is convenient in some cases to attribute to the object-glass the central aberration of a greater amount and so compensate by it the aberration of the eyepiece. In the theory of the aberrations of the telescopic systems there are many other problems not even mentioned in this paper. I quite agree with Mr Smith that the theory which neglects the aberrations of the eyepiece completely is not of general value, but I must note that we had no intention at all to give here any *theory* of aberrations of the telescopic systems.

ON SYSTEMS OF PLANE REFLECTING SURFACES

By T. SMITH, M.A., F.Inst.P.

(Optics Department, The National Physical Laboratory.)

MS. received, 25*th July*, 1928. *Read and discussed*, 8*th November*, 1928.

ABSTRACT. An algebraic method is evolved of finding the coordinates of the image of any point and of the direction of the emergent portion of any given incident ray after reflection at any number of plane reflecting surfaces. Systems of reflectors are classified according to the nature of the self-conjugate region of the field. A method of designing a system having any assigned properties is described. Suitable criteria are given to determine whether with a prismatic system the whole is non-dispersive, and whether total internal reflection takes place at any given surface; also the boundary conditions at each surface are found. The calculations are simple and free from any ambiguity of sign.

The general algebraic theory

Systems of plane reflectors, usually arranged to deflect light while it is within a block of glass, are of considerable importance in certain types of optical instrument. The properties of such systems can readily be exhibited graphically when the normals to the various surfaces are coplanar, but in more general examples graphical methods are apt to prove troublesome. The usual method of investigation is then to calculate the properties of the system by spherical trigonometry. In the endeavour to break away from this method Captain Baker has lately resorted to drawing on a sphere. The details of his procedure, in so far as they relate to the directions of the rays, have been described in a paper read before this Society*. For a rapid preliminary survey of the types of construction which may be employed to secure a given result this method has much to recommend it, but, like all graphical methods, it can hardly be expected to yield the accuracy essential for optical work. Ultimately the alternative to a trigonometrical is an algebraic method, such as we now proceed to describe.

Whatever ρ may be $(x + \rho l,\ y + \rho m,\ z + \rho n)$ is a point on the straight line through (x, y, z) normal to the plane

$$p - lx - my - nz = 0 \qquad \ldots\ldots(1).$$

The first-mentioned point lies in the plane if $\rho = p - lx - my - nz$, and since this point bisects the straight line joining (x, y, z) to its image by reflection in (1), it follows that the image point is $(x + Pl,\ y + Pm,\ z + Pn)$, where

$$P = 2\,(p - lx - my - nz).$$

Let us now suppose that we have k planes

$$p_j - l_j x - m_j y - n_j z = 0 \quad (j = 1, 2, 3, \ldots k),$$

* T. Y. Baker, *Trans. Opt. Soc.* **29** 1927–8 (49).

which are met in this order by the light. Write

$$\widehat{ij} = -2(l_il_j + m_im_j + n_in_j).$$

It follows by repeated substitution that the equations giving the coordinates (x', y', z') of the image of (x, y, z), written at length, are

$$\begin{aligned} x' = {} & x + P_1l_1 + (P_2 + P_1\widehat{12})\,l_2 + (P_3 + P_2\widehat{23} + P_1\widehat{13} + P_1\widehat{1223})\,l_3 \\ & + (P_4 + P_3\widehat{34} + P_1\widehat{24} + P_2\widehat{2334} + P_1\widehat{14} + P_1\widehat{1224} + P_1\widehat{1334} + P_1\widehat{122334})\,l_4 \\ & + \ldots \qquad \ldots\ldots(2), \end{aligned}$$

with two similar equations having y and m, and z and n respectively in place of x and l. The law of formation of the terms is obvious. For a small number of reflections such expressions may be serviceable, but in general it is better to conform more closely to the procedure followed in the kindred theory of curved surfaces and consider the law of formation of the coefficients as a whole. For brevity we will denote the matrix*

$$\begin{pmatrix} 1 - 2l^2 & -2lm & -2ln & 0 \\ -2lm & 1 - 2m^2 & -2mn & 0 \\ -2ln & -2mn & 1 - 2n^2 & 0 \\ 2lp & 2mp & 2np & 1 \end{pmatrix} \qquad \ldots\ldots(3)$$

by (R), and will substitute (R_j) for (R) when l, m, n, p carry the suffix j. For reflection at a single surface (2) and the two similar equations are equivalent to the matrix equation

$$(x\ y\ z\ 1)\,(R) = (x'\ y'\ z'\ 1) \qquad \ldots\ldots(4)\dagger.$$

It at once follows that after k reflections the same equation holds provided (R_{1k}) is written for (R) where

$$(R_{1k}) = (R_1)\,(R_2)\,(R_3) \ldots (R_k) \qquad \ldots\ldots(5).$$

Conversely if we interchange the accented and unaccented letters in (5), (R) must be replaced by (R_{k1}) where

$$(R_{k1}) = (R_k)\,(R_{k-1})\,(R_{k-2}) \ldots (R_1) \qquad \ldots\ldots(6).$$

It is easily seen that $(R_j)^2 = 1$‡, and therefore

$$(R_{1k})\,(R_{k1}) = 1 = (R_{k1})\,(R_{1k}) \qquad \ldots\ldots(7).$$

Written at length these matrices take the form

$$(R_{1k}) = \begin{pmatrix} a & d & g & 0 \\ b & e & h & 0 \\ c & f & i & 0 \\ \xi' & \eta' & \zeta' & 1 \end{pmatrix},$$

$$(R_{k1}) = \begin{pmatrix} a & b & c & 0 \\ d & e & f & 0 \\ g & h & i & 0 \\ \xi & \eta & \zeta & 1 \end{pmatrix},$$

* A short introduction to the properties of matrices has been given in *Trans. Opt. Soc.* **29** 1927–8 (168).

† For the meaning of a linear matrix see *Trans. Opt. Soc.* **29** 1927–8 (177).

‡ Obviously $|R_j|$ is negative, and therefore $|R_j| = -1$.

where the letters in the first three rows are the direction cosines of three mutually orthogonal lines. From (4) we see that (ξ', η', ζ') is the image of the origin after the reflections 1, 2, 3, ... k, and similarly (ξ, η, ζ) is the image of the origin after the reflections $k, k-1, \ldots 2, 1$.

By regarding (x, y, z) as the end of a unit vector drawn from the origin, we deduce that the direction cosines $a, b, \ldots i$ determine completely the directions of all emergent rays in terms of their incident directions and vice versa. In other words the matrix (R) contains the complete solution of the problem.

As an example consider the reflection of light at the three orthogonal planes

$$x = X,\ y = Y,\ z = Z.$$

The three component matrices are

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2X & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2Y & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 2Z & 1 \end{pmatrix},$$

and their product in any order gives the resultant

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 2X & 2Y & 2Z & 1 \end{pmatrix},$$

showing that the direction of every ray is reversed, and that the image of (x, y, z) is $(2X - x, 2Y - y, 2Z - z)$.

The condition for no dispersion

When it is only necessary to consider the directions of the rays, the last row and the last column of each matrix may be omitted, so that the matrices fall to the third order. Such matrices may be distinguished by the use of lower-case letters instead of capitals. For example, if light, after refraction into a prism at a plane surface having the direction (l, m, n) and subsequent deviation at a number of reflecting surfaces, is to be refracted out without dispersion, so that the prism is equivalent to a plane parallel plate of glass, the direction cosines (l', m', n') of the final refracting surface are given by the equation

$$(l\ m\ n)\,(r) = (l'\ m'\ n').$$

The condition that the prism is equivalent to a parallel plate may be expressed in the form

$$(l\ m\ n)\,(r)\begin{pmatrix} l' \\ m' \\ n' \end{pmatrix} = 1 \qquad \ldots\ldots(8).$$

The condition for total internal reflection

It will often be necessary to determine whether total internal reflection occurs at a particular surface j. The condition for such reflection may be expressed in the form that the cosine of the angle of incidence in the prism does not exceed a certain value κ, depending on the refractive indices of the material of the prism and of the medium on the other side of the surface in question. The condition to be satisfied is therefore

$$(L\ M\ N)\,(r_{1j-1})\begin{pmatrix} l_j \\ m_j \\ n_j \end{pmatrix} \gtrless \kappa \qquad \text{.....(9),}$$

where (L, M, N) is the direction in which the ray is incident on the first reflecting surface. As the equation of any surface is unaltered if the signs of l, m, n, p are all altered, the positive sign should always be given to the product on the left side of equation (9). Since the product is merely changed in sign if (r_{1j-1}) is replaced by (r_{1j}) in (9), we may, with this understanding regarding the signs, use either of these matrices as is found convenient.

The boundary conditions

Before we can say that a prism made with surfaces having given equations is suitable for the purpose for which it is suggested, it is necessary to be sure that light incident through given points in specified directions will be transmitted and will emerge in the directions desired. Both these conditions are reducible to the determination of the order in which a ray meets the various surfaces of the prism. Let us suppose that the ray which is originally incident through (x, y, z) in the direction (L, M, N) has been found to meet the surfaces $1, 2, \ldots i$ in the given order; we wish to find which surface will be met next. The point on the incident ray conjugate to the point of reflection at surface i is of the form

$$(x + \rho_i L, y + \rho_i M, z + \rho_i N),$$

and similarly that of the next surface j will be $(x + \rho_j L, y + \rho_j M, z + \rho_j N)$. Evidently we must have $\rho_j > \rho_i$, but subject to this limitation the correct value of j is that for which $\rho_j - \rho_i$ is least. In the exceptional case of a prism with re-entrant angles, it is further necessary to determine the actual coordinates of the point of reflection when a surface bordering a re-entrant angle is indicated by the criterion just given. By (4) the ρ's are to be found from the equation

$$\rho_j\,(L\ M\ N)\,(r_{1i})\begin{pmatrix} l_j \\ m_j \\ n_j \end{pmatrix} + (x\ y\ z\ 1)\,(R_{1i})\begin{pmatrix} l_j \\ m_j \\ n_j \\ -p_j \end{pmatrix} = 0 \qquad \text{.....(10).}$$

This equation is also suitable for finding the length of path within a prism between entrance and emergence. The condition $\rho_j > \rho_i$ must generally be satisfied for all

rays transmitted by the system. This involves no more than the determination of a limiting value for p_j for one particular bounding ray which is usually selected without difficulty.

The boundary conditions can easily be treated in other ways. For example, suppose after reflection at the surface i the light next meets surface j, and it is desired to know whether this is suitably placed for its intersection with surface k not to encroach within the light cone. The coordinates of a point on the intersection of surfaces j and k are given by

$$\frac{x'-q\lambda}{p_j(l_j-l_k\kappa)+p_k(l_k-l_j\kappa)}=\frac{y'-q\mu}{p_j(m_j-m_k\kappa)+p_k(m_k-m_j\kappa)}$$
$$=\frac{z'-q\nu}{p_j(n_j-n_k\kappa)+p_k(n_k-n_j\kappa)}=\frac{1}{1-\kappa^2},$$

where q may have any value and

$$\frac{\kappa}{l_jl_k+m_jm_k+n_jn_k}=\frac{\lambda}{m_jn_k-m_kn_j}=\frac{\mu}{n_jl_k-n_kl_j}=\frac{\nu}{l_jm_k-l_km_j}=1.$$

The coordinates corresponding to (x', y', z') in the initial space are found as functions of q from

$$(x\ y\ z\ 1)=(x'\ y'\ z'\ 1)\,(R_{(i)}).$$

If now the incident pencil is bounded by the cone $f\left(\frac{y-y_0}{x-x_0}, \frac{z-z_0}{x-x_0}\right)=0$, and we substitute in this equation the values of x, y, z we have just found, the line in which the planes j and k meet will intersect the cone if the equation is satisfied for real values of q; the intersections will be imaginary if the roots are imaginary. When equal roots occur the line of intersection is in a limiting position. It is advantageous to consider the values of p_j which yield equal roots when it is important to make the prism as small as possible. In general two values will be found for p_j, of which one corresponds to the case when light meets surface k instead of surface j after reflection at i. There is no difficulty in formulating a test to discriminate between these two values when this method is followed.

The classification of reflecting systems

We have already noticed that for a single surface $|r|=|R|=-1$. The corresponding determinants for a succession of reflecting surfaces are obtained by multiplying together the determinants for each separate reflection. It follows that

$|r|=-1$ if the number of reflections is odd

and $|r|=1$ if the number of reflections is even.

On taking into account the orthogonal properties of r, we can at once see that

$$T=t\,|r|,$$

where t denotes any element of (r) and T is the minor of t in $|r|$. If now

$$1=\pm u=\pm v=\pm w,$$

we have

$$\begin{vmatrix} a+u & b & c \\ d & e+v & f \\ g & h & i+w \end{vmatrix} = uvw + avw + ewu + iuv + Au + Ev + Iw + |r|$$

$$= (uvw + |r|)(1 + au + ev + iw) \qquad \text{.....(11).}$$

The significance of this relation will be apparent on referring to (4), which shows that

$$\frac{x}{\Delta_1} = \frac{-y}{\Delta_2} = \frac{z}{\Delta_3} = \frac{-1}{\Delta_4} \qquad \text{.....(12),}$$

where $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ are the determinants obtained by omitting the first, second, third, fourth columns respectively of the array

$$\left\| \begin{matrix} a+u & b & c & \xi' - x' - ux \\ d & e+v & f & \eta' - y' - vy \\ g & h & i+w & \zeta' - z' - wz \end{matrix} \right\| \qquad \text{.....(13).}$$

Thus we find the points which are self-conjugate under this transformation by putting

$$u = v = w = -1.$$

The primary division it is natural to make is that between systems in which the total number of reflections is even and those in which it is odd. These main divisions may be divided again according to the nature of the self-conjugate locus. This locus may consist of

(α) the whole three-dimensional field,
(β) a plane,
(γ) a straight line,
(δ) a point,
(ϵ) no point at a finite distance.

We consider each of these classes briefly in turn.

(α) The exact superposition of the whole field obviously involves $(R) = 1$, which includes $(r) = 1$. The number of reflections is therefore even. For practical purposes this class is unimportant, as we can only satisfy the conditions $\xi' = \eta' = \zeta' = 0$ in a system of vanishingly small dimensions.

The characteristic of the α class is that every element of $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ vanishes.

(β) A plane locus will only be secured when the three conditions

$$x' - x = y' - y = z' - z = 0$$

lead to the same relation between x, y, z, or else to identities. The case when all three conditions are identically satisfied is here excluded, for it obviously belongs to class α. We may therefore write the array (13) in the form

$$\left\| \begin{matrix} \lambda l & \lambda m & \lambda n & -\lambda p \\ \mu l & \mu m & \mu n & -\mu p \\ \nu l & \nu m & \nu n & -\nu p \end{matrix} \right\|,$$

where at least one of the three quantities λ, μ, ν is not zero, and similarly one at least of l, m, n is not zero. By multiplying throughout by suitable constants we can impose the conditions

$$\lambda^2 + \mu^2 + \nu^2 = l^2 + m^2 + n^2 = 1.$$

From the relations of the coefficients in the first three columns of the array to a mutually orthogonal system of direction cosines we obtain equations such as

$$(\lambda l + 1)^2 + \lambda^2 m^2 + \lambda^2 n^2 = 1,$$

$$(\lambda l + 1)\,\mu l + \lambda m\,(\mu m + 1) + \lambda n \mu n = 0,$$

which involve $\lambda = 0$ or $-2l$, $\lambda\mu + \lambda m + \mu l = 0$.

It at once follows that the matrix is identical with (3), or the transformation can be effected by reflection at a single plane. The number of reflections involved when the self-conjugate locus is a plane is thus necessarily odd.

Equation (11) shows that Δ_4 vanishes for members of this class because $a + e + i = 1$.

The characteristic of the β class is that every first minor of Δ_1, Δ_2, Δ_3, Δ_4 vanishes.

(γ) When the self-conjugate locus is a straight line there is an identical relation between the three conditions $x' - x = y' - y = z' - z = 0$. Let it be

$$\lambda\,(x' - x) + \mu\,(y' - y) + \nu\,(z' - z) = 0.$$

Then

$$\begin{aligned} \lambda\,(a - 1) + \mu d + \nu g &= 0, \\ \lambda b + \mu\,(e - 1) + \nu h &= 0, \\ \lambda c + \mu f + \nu\,(i - 1) &= 0, \\ \lambda\xi' + \mu\eta' + \nu\zeta' &= 0. \end{aligned}$$

Multiplying the first three in turn by a, b, c and adding, then by c, d, e, and lastly by f, g, h, we obtain the additional equations

$$\begin{aligned} \lambda\,(a - 1) + \mu b + \nu c &= 0, \\ \lambda d + \mu\,(e - 1) + \nu f &= 0, \\ \lambda g + \mu h + \nu\,(i - 1) &= 0. \end{aligned}$$

Subtracting the first equations of each set, and similarly dealing with the second and third pairs, we obtain

$$\frac{h - f}{\lambda} = \frac{c - g}{\mu} = \frac{d - b}{\nu} = 2t, \text{ say.}$$

But from the orthogonal relations we find

$$h^2 - f^2 = c^2 - g^2 = d^2 - b^2,$$

so that we may put

$$\lambda\,(h + f) = \mu\,(c + g) = \nu\,(d + b) = 2\lambda\mu\nu s, \text{ say.}$$

We can therefore express b, c, d, f, g, h in terms of λ, μ, ν, and by substituting in the original equations a, e, i are obtained as functions of the same variables. In fact

$$a = 1 + (\lambda^2 - 1)\,s, \quad b = \lambda\mu s - \nu t, \quad c = \nu\lambda s + \mu t,$$

and since we may evidently assume $\lambda^2 + \mu^2 + \nu^2 = 1$, the identity $a^2 + b^2 + c^2 = 1$ gives $(1 - s)^2 + t^2 = 1$, or $s = 1 - \cos\theta$, $t = \sin\theta$. It is now easy to verify that $|r| = 1$, $a + e + i = 1 + 2\cos\theta$, and that the transformation denotes a rotation about the direction (λ, μ, ν) through the angle θ. The number of reflections is always even.

The characteristic of the γ class is that $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ all vanish, but non-zero minors exist.

(δ) Since for this class Δ_4 must not vanish, the number of reflections is necessarily odd. A single reflection is an example of class β, so that at least three reflections are necessary to attain a transformation of this class.

The characteristic of the δ class is that Δ_4 is not zero.

(ϵ) The four classes that have been considered are exhaustive apart from the possibility of two or more of the conditions being inconsistent with one another. Such inconsistency can only arise when $\Delta_4 = 0$, and the present group thus includes three distinct types having Δ_4 of the character found in classes α, β, γ respectively. We will denote these by $\alpha^*, \beta^*, \gamma^*$.

(α^*) In this class the axes of x', y', z' are parallel to those of x, y, z but not all coincident with them. We can construct members of this class by adding to any system a directionally equivalent system with the latter surfaces displaced so as to be encountered in the reverse order. Thus

$$\begin{pmatrix} a & d & g & 0 \\ b & e & h & 0 \\ c & f & i & 0 \\ \xi_1 & \eta_1 & \zeta_1 & 1 \end{pmatrix}\begin{pmatrix} a & b & c & 0 \\ d & e & f & 0 \\ g & h & i & 0 \\ \xi_2 & \eta_2 & \zeta_2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ X & Y & Z & 1 \end{pmatrix},$$

where $$X = a\xi_1 + d\eta_1 + g\zeta_1 + \xi_2, \text{ etc.}$$

Clearly we may have any even number of reflections in a system of this class. The combination of an α^* system with any other system will not change the class letter of the latter, but it may involve the addition or the deletion of an asterisk. It follows that apart from a single surface, which is necessarily a member of class β, and two surfaces, which yield either a γ or an α^* system, the number of surfaces of any class is not restricted other than by its being even or odd. The matrix product exhibiting these results is

$$\begin{pmatrix} a & d & g & 0 \\ b & e & h & 0 \\ c & f & i & 0 \\ \xi & \eta & \zeta & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ X & Y & Z & 1 \end{pmatrix} = \begin{pmatrix} a & d & g & 0 \\ b & e & h & 0 \\ c & f & i & 0 \\ \xi + X & \eta + Y & \zeta + Z & 1 \end{pmatrix}.$$

The characteristic of the α^* class is that every element of Δ_4 vanishes, but not every element of $\Delta_1, \Delta_2, \Delta_3$.

(β^*) Members of this class have the emergent axes displaced relatively to the incident axes in a direction parallel to the equivalent plane. $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ all vanish, but while all the first minors of Δ_4 vanish there is at least one non-vanishing

minor in the remaining determinants. Since by combining with a suitable α^* system we can add any desired X, Y, Z to the ξ, η, ζ of the system, we can secure such values for the sums as will cause all the first minors to vanish.

The characteristic of the β^* class is that every first minor of Δ_4 vanishes but not every first minor of Δ_1, Δ_2, Δ_3.

(γ^*) In this class the emergent points are found by applying a constant screw displacement to the incident points. Members may be constructed by adding an α^* system to a γ system.

The characteristic of the γ^* class is that Δ_4 vanishes but Δ_1, Δ_2, Δ_3 are not all zero.

The solution of the matrix equations

The problem which confronts the designer is more frequently the determination of a system of surfaces which will produce a required transformation than finding the effect of a given system. The only part of the problem which can occasion any trouble is that concerning the directions of the rays, that is to say, we need only show how to find a system which has a given (r). The solutions for the α and β classes have already been given. We obtain a member of the γ class by taking two intersecting planes. Let their direction cosines be (l_1, m_1, n_1) and (l_2, m_2, n_2), and let them meet at the angle θ in a straight line having the direction

$$(\lambda \operatorname{cosec}\theta,\ \mu \operatorname{cosec}\theta,\ \nu \operatorname{cosec}\theta).$$

Then if $\cos\theta = \kappa$,

$$l_1l_2 + m_1m_2 + n_1n_2 = \kappa \qquad \text{.....(14)},$$

$$\frac{m_1n_2 - m_2n_1}{\lambda} = \frac{n_1l_2 - n_2l_1}{\mu} = \frac{l_1m_2 - l_2m_1}{\nu} = 1 \qquad \text{.....(15)},$$

$$\kappa^2 + \lambda^2 + \mu^2 + \nu^2 = 1 \qquad \text{.....(16)},$$

and on applying (5) to the two matrices of pattern (3) we find

$$(r_{12}) = \begin{pmatrix} 2(\kappa^2+\lambda^2) - 1 & 2(\lambda\mu + \nu\kappa) & 2(\lambda\nu - \mu\kappa) \\ 2(\lambda\mu - \nu\kappa) & 2(\kappa^2+\mu^2) - 1 & 2(\mu\nu + \lambda\kappa) \\ 2(\lambda\nu + \mu\kappa) & 2(\mu\nu - \lambda\kappa) & 2(\kappa^2+\nu^2) - 1 \end{pmatrix} \qquad \text{.....(17)},$$

in agreement with the characteristic formula of type (γ). It follows that any transformation of this class may be represented by a system of two planes, and that it is specified by the four variables κ, λ, μ, ν.

We have now to show how to construct a reflecting system corresponding to any given matrix of class δ. Let us enclose in matrix brackets the letters β, γ, δ to indicate the directional matrices of a member of the corresponding classes. Then we have solved the problem if we can resolve (δ) into either of the forms (β) (γ) or (γ) (β). Taking the latter and multiplying behind by (β) the equation (δ) = (γ) (β) becomes (δ) (β) = (γ). But since (γ) is the general form of matrix for an even number of reflections, the choice of (β) is arbitrary. We therefore find the general solution of this class of system by adding any plane through the self-conjugate point and finding the axis and amount of rotation needed to complete the transformation.

Practical calculations

It will be apparent from the results just obtained that an unnecessary amount of work is involved in calculating the nine elements of the directional matrix as only four elements are in fact needed. Let us return to the matrix (R) with ξ', η', ζ' all zero and the directional constants of the form found in (17) except that every element is made homogeneous of the second order in κ, λ, μ, ν by (16). Then (R) is the square of the matrix

$$\begin{pmatrix} \kappa & \nu & -\mu & \lambda \\ -\nu & \kappa & \lambda & \mu \\ \mu & -\lambda & \kappa & \nu \\ \lambda & \mu & \nu & -\kappa \end{pmatrix}.$$

Let (s) denote the matrix obtained by changing the signs of every element in the last column, so that

$$(s) = \begin{pmatrix} \kappa & \nu & -\mu & -\lambda \\ -\nu & \kappa & \lambda & -\mu \\ \mu & -\lambda & \kappa & -\nu \\ \lambda & \mu & \nu & \kappa \end{pmatrix} \qquad \text{.....(18).}$$

We can show that on combining matrices of this form their character is unchanged, and the coefficients of reflection are obtained by multiplying such matrices, that is,

$$(s_{1k}) = (s_1)(s_2)(s_3) \ldots (s_k) \qquad \text{.....(19).}$$

Since the matrix involves each constant four times, only one line is evaluated in practice, the rest if needed being obtained from (18).

We have now only to notice that this expression is applicable to the calculation of every case that exists, including those in which the number of surfaces is odd. To see this we have only to observe that (17) in the form in which we have been using it is identical with (3) if we put $(\kappa\, \lambda\, \mu\, \nu) = (o\, il\, im\, in)$, where ι denotes the square root of -1. For an odd number of reflections every element of (s) is thus imaginary. Equation (16) then assumes the more general form

$$\kappa^2 + \lambda^2 + \mu^2 + \nu^2 = |r|.$$

When (r) is symmetrical, $\kappa = o$, and the derivation of λ, μ, ν is simple. In all other cases the equations

$$\frac{1}{4\kappa} = \frac{\kappa}{a+e+i+|r|} = \frac{\lambda}{h-f} = \frac{\mu}{c-g} = \frac{\nu}{d-b} \qquad \text{.....(20)}$$

may be used.

The other matrices of the r type that have been mentioned earlier in this paper may advantageously be replaced by matrices of the s type in actual calculations. It will suffice to take as an example the quantity occurring in equations (8), (9), and

(10), which measures the sine of the angle made by a ray with a surface. Let (P) denote the s matrix when $(\kappa\ \lambda\ \mu\ \nu) = (0\ L\ M\ N)$, and similarly (p) the s matrix with elements $(0\ l\ m\ n)$. Then on multiplying out we find

$$(L\ M\ N)(r)\begin{pmatrix} l \\ m \\ n \end{pmatrix}$$

$$= (\lambda L + \mu M + \nu N)(\lambda l + \mu m + \nu n) + (\kappa L - \nu M + \mu N)(\kappa l + \nu m - \mu n)$$
$$+ (\nu L + \kappa M - \lambda N)(-\nu l + \kappa m + \lambda n) + (-\mu L + \lambda M + \kappa N)(\mu l - \lambda m + \kappa n),$$

that is to say, the cosine of the angle of incidence is the κ of any of the s matrices $(Ptpt')$, $(tpt'P)$, $(pt'Pt)$, $(t'Ptp)$, where (t) is the s matrix corresponding to the system intervening between the initial space and the given surface, and (t') denotes the same system taken in the reverse direction.

Conclusion

From the foregoing examination it is apparent that systems of reflectors may be either designed or analysed rapidly and with little trouble by the algebraic method, which is free from difficulties arising from uncertainty in the sign of angles such as have been found in trigonometrical methods of investigating these systems.

For Discussion see p. 94.

REFLECTING SYSTEMS FOR IMAGE INVERSION

By T. SMITH, M.A., F.Inst.P.

(Optics Department, The National Physical Laboratory.)

MS. received, 13th September, 1928. Read and discussed, 8th November, 1928.

ABSTRACT. The method of investigating systems of plane reflectors described in another paper* has been applied to determine how many optical surfaces are necessary in an inverting prism. Four surfaces involve oblique refraction into the prism whatever the number and order of the reflections. With five surfaces one form is possible with four reflections. All possible arrangements with six reflections at five surfaces are considered, and the application of the method to prisms with a greater number of reflections is illustrated.

In the previous paper* I have given an account of an algebraic method of treating the problems concerning the use of series of plane reflecting surfaces with which designers of optical instruments are often faced. It has recently been suggested that the last word may not yet have been said on reflecting systems for the inversion of an image such as are used in prismatic binoculars; in particular the number of optically worked surfaces required may be susceptible of reduction if one surface or more may serve in a double capacity. I propose here to examine this question with the aid of the method mentioned.

In the preceding paper the only operation involved not familiar to everyone is the multiplication of matrices. This process was explained in an earlier paper†. In this investigation the simpler processes of addition and subtraction are also involved, and a few words of explanation on these are offered.

The symbols (a), (b), (c) are convenient abbreviations for

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \cdots \\ a_{21} & a_{22} & a_{23} \cdots \\ a_{31} & a_{32} & a_{33} \cdots \\ \cdots\cdots\cdots\cdots \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} & b_{13} \cdots \\ b_{21} & b_{22} & b_{23} \cdots \\ b_{31} & b_{32} & b_{33} \cdots \\ \cdots\cdots\cdots\cdots \end{pmatrix}, \begin{pmatrix} c_{11} & c_{12} & c_{13} \cdots \\ c_{21} & c_{22} & c_{23} \cdots \\ c_{31} & c_{32} & c_{33} \cdots \\ \cdots\cdots\cdots\cdots \end{pmatrix}.$$

We define (c) as the sum of (a) and (b) if

$$c_{11} = a_{11} + b_{11},\ c_{12} = a_{12} + b_{12},\ c_{13} = a_{13} + b_{13}, \ldots$$
$$c_{21} = a_{21} + b_{21},\ c_{22} = a_{22} + b_{22},\ c_{23} = a_{23} + b_{23}, \ldots$$

and so on. It follows that $(a) + (b) = (c) = (b) + (a)$, so that the order of the terms in an expression may be altered exactly as in ordinary algebra. The extension of the addition law to any number of terms is obvious. When we add together k

* "On Systems of Plane Reflecting Surfaces," *Trans. Opt. Soc.* **30** 1928–9 (68).

† *Trans. Opt. Soc.* **29** 1927–8 (168).

matrices each identical with (a), we are led, if we wish to write the sum of (a) repeated k times as $k\,(a)$, to the relation

$$\begin{pmatrix} ka_{11} & ka_{12} & ka_{13} \dots \\ ka_{21} & ka_{22} & ka_{23} \dots \\ \dots\dots\dots\dots\dots \end{pmatrix} = k\,(a)$$

previously noted. This is assumed to hold when k is no longer a positive integer, and in the particular case when $k = 0$ justifies our regarding (a) itself as having the value 0 when each of its elements vanishes. If we put $k = -1$ we are led to

$$\begin{array}{lll} c_{11} = a_{11} - b_{11} & c_{12} = a_{12} - b_{12} & c_{13} = a_{13} - b_{13} \\ c_{21} = a_{21} - b_{21} & c_{22} = a_{22} - b_{22} & c_{23} = a_{23} - b_{23} \\ \dots\dots\dots\dots & \dots\dots\dots\dots & \dots\dots\dots\dots \end{array}$$

as the meaning of $(c) = (a) - (b)$. In such a case it is obvious that $(a) = (b) + (c)$, so that an expression containing matrices may be simplified in the same way as in ordinary algebra if only addition and subtraction are involved. We also see that $(a) - (b) = 0$ or $(a) = (b)$ implies $a_{pq} = b_{pq}$ whatever values may be given to p and q.

For reasons given when the multiplication of matrices was considered the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \dots \\ 0 & 1 & 0 & 0 \dots \\ 0 & 0 & 1 & 0 \dots \\ 0 & 0 & 0 & 1 \dots \\ \dots\dots\dots\dots\dots \end{pmatrix},$$

in which all the elements of the leading diagonal are equal to unity and all the others are zero, is considered itself to have the value unity. It follows that an equation such as

$$\alpha\,(a) + \beta\,(b) + \gamma\,(c) + \delta = 0$$

is to be considered the equivalent of the equations

$$\begin{array}{llll} \alpha a_{11} + \beta b_{11} + \gamma c_{11} + \delta = 0, & \alpha a_{12} + \beta b_{12} + \gamma c_{12} = 0, & \alpha a_{13} + \beta b_{13} + \gamma c_{13} = 0, \dots \\ \alpha a_{21} + \beta b_{21} + \gamma c_{21} = 0, & \alpha a_{22} + \beta b_{22} + \gamma c_{22} + \delta = 0, & \alpha a_{23} + \beta b_{23} + \gamma c_{23} = 0, \dots \end{array}$$

and so on, the δ only appearing in those equations in which the two suffixes carried by a, b, c are equal.

Two matrices (a) and (b) are said to be conjugate if $b_{qp} = A_{pq}/\Delta$, where A_{pq} is the minor of a_{pq} in the determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \dots \\ a_{21} & a_{22} & a_{23} \dots \\ a_{31} & a_{32} & a_{33} \dots \\ \dots\dots\dots\dots\dots \end{vmatrix} = \Delta.$$

From well-known properties of the determinant it follows that

$$(a)\,(b) = (b)\,(a) = 1.$$

In the matrices which occur in this investigation—the s matrices of the previous paper—the elements of the leading diagonal are all equal, and the elements symmetrically situated with respect to the diagonal are equal in value but opposite in sign. Moreover, the matrix is converted into its conjugate by the interchange of rows and columns. In symbols, if (a) and (b) are conjugate,

$$a_{pp} = b_{pp} = a_{qq} = b_{qq}, \; a_{pq} = -a_{qp} = -b_{pq} = b_{qp},$$

where p and q represent any different integers. It follows that

$$(a) + (b) = 2a_{11}.$$

With the exception of one equation which rests on the elementary properties of determinants, this investigation involves nothing more than simple applications of the properties of matrices which have now been explained and the meanings of the symbols $\kappa, \lambda, \mu, \nu$ given in the previous paper.

Prism with two reflections

Taking the x-axis as the axis of the objective, the basic problem is the construction of a system of surfaces having the directional matrix

$$(r) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

inspection showing that an even number of reflections is involved. The transformation is of type γ, the characteristic constants being

$$(\kappa \; \lambda \; \mu \; \nu) = (0 \; 1 \; 0 \; 0).$$

The simplest solution therefore consists of two reflectors at right angles to one another with their line of intersection parallel to the axis of the lens. This system is open to the practical objection that light would have to enter the prism at oblique incidence to ensure deviation towards the reflecting surfaces. This would tend to restrict the field of view to a comparatively small size. The prism, which is merely the erecting prism used in projecting experiments on a screen, with a roof in place of the plane base, is chiefly of interest as an illustration that the customary six surfaces can be reduced to four. It is illustrated in Fig. 1 (p. 88).

Prisms with four surfaces

Let us now consider systems with four or more reflections when the light enters and leaves the prism at normal incidence. This means that we have in the prism two surfaces perpendicular to the x-axis. Denote the directions of these surfaces by the figure 0, and other surfaces in different directions by 1, 2, etc. If we suppose that only four surfaces in all are needed, the reflection constants of the system can be expressed as sums in a way which enables us to deal very comprehensively with all the systems that may be proposed. This arises from the fact that in any group of four successive surfaces at least one direction must occur twice.

We have seen in equation (19) of the previous paper that the directional constants of any system 1, 2, 3, ... k, are calculable from the equation

$$(s_{123\ldots k}) = (s_1)(s_2)(s_3)\ldots(s_k).$$

Now for any single surface $(s)^2 = 1$, so if any surface is followed immediately by another in the same direction both may be deleted: thus $(1002) = (12)$*. Again from equation (18), for a system consisting of an even number of surfaces

$$(s) + (s') = 2\kappa \qquad \ldots\ldots(1),$$

where (s') is the matrix conjugate to (s), and so corresponds to the system in which the surfaces are encountered in the reverse order. If we write bc for s, b and c being any two parts into which the system can be divided, we must write $c'b'$ for s', so that

$$(bc) + (c'b') = 2\kappa_{bc}.$$

Multiply by (a) in front and (bd) behind. Then

$$(abcbd) + (ac'd) = 2\kappa_{bc}(abd) \qquad \ldots\ldots(2).$$

This enables us to simplify a matrix in which a number of surfaces b is repeated after the intervention of the set c provided the number in sets b and c together is even. For example $(1012) = -(02) + 2\kappa_{10}(12)$. Again from (1) if both b and c represent even sets of surfaces

$$(bc) + (bc') = 2\kappa_c(b),$$
$$(cb) + (cb') = 2\kappa_b(c),$$
$$(bc') + (cb') = 2\kappa_{bc'} = 2\{\kappa_b\kappa_c + \lambda_b\lambda_c + \mu_b\mu_c + \nu_b\nu_c\},$$

and therefore $\qquad (bc) + (cb) = 2\{\kappa_b(c) + \kappa_c(b) - \kappa_{bc'}\},$

and $\qquad (abcd) + (acbd) = 2\{\kappa_b(acd) + \kappa_c(abd) - \kappa_{bc'}(ad)\} \qquad \ldots\ldots(3).$

This transposition formula enables us to reduce matrices to which (2) is inapplicable. For instance, if $(bc) = (1021)$, $(cb) = (2110) = (20)$.

By means of these formulae it is clear that we can reduce any s matrix to the form

$$(s) = \alpha(X) + \beta(Y) + \gamma(Z) + \delta \qquad \ldots\ldots(4),$$

where $(X), (Y), (Z)$ denote the matrices for the combinations 21, 10, 02 respectively. More generally any system reducing to a cyclic combination such as

$$(X) = (bc'),\ (Y) = (ca'),\ (Z) = (ab'),$$

may be represented by this formula.

Now four cases may arise. In the first α, β, γ are all zero. Then $(s) = 1$, that is to say, the system is not inverting. In the second only one of these coefficients is not zero, and the equation reduces to $(s) = (X)$ or $(s) = (X')$. The axis of X is thus parallel to that about which rotation of the field is to be effected, and when X is obtained from two surfaces each of these is parallel to the s-axis. Such systems are without interest for the problem we are discussing. In the third case only one of these coefficients vanishes: let us suppose that α and β remain. The equation then indicates that the axes of s, X, and Y lie parallel to a single plane, which is clearly

* Where no uncertainty can be caused the s may be omitted and the suffix put in ordinary type.

the plane involved in both X and Y in our prismatic problem. Now the surface o is normal to s, and thus either 1 or 2 must be parallel to s. Moreover, the first and last reflections take place at different surfaces, and neither of these is in the direction o. Systems belonging to this third case may therefore be rejected. We have then only to consider those systems in which none of the coefficients α, β, γ vanishes. If Δ denote the determinant

$$\begin{vmatrix} \kappa & \lambda & \mu & \nu \\ \kappa_X & \lambda_X & \mu_X & \nu_X \\ \kappa_Y & \lambda_Y & \mu_Y & \nu_Y \\ \kappa_Z & \lambda_Z & \mu_Z & \nu_Z \end{vmatrix}$$

and K, K_X, K_Y, K_Z the minors of $\kappa, \kappa_X, \kappa_Y, \kappa_Z$, the matrix equation is obviously

$$\Delta = K(s) + K_X(X) + K_Y(Y) + K_Z(Z).$$

Now multiply before and behind by (X). All the terms can be simplified by (2). In particular

$$(XYX) = -(Y') + 2\kappa_Z(X) = (Y) + 2\kappa_Z(X) - 2\kappa_Y$$

since $$(XY) = (bc'ca') = (ba') = (Z').$$

Thus we find

$$\Delta\{2\kappa_X(X) - 1\} = K\{2\kappa_{sX}(X) - (s)\} + K_X\{(4\kappa_X^2 - 1)(X) - 2\kappa_X\} + K_Y\{(Y) + 2\kappa_Z(X) - 2\kappa_Y\} + K_Z\{(Z) + 2\kappa_Y(X) - 2\kappa_Z\}$$

which simplifies at once to

$$\{K(\lambda\lambda_X + \mu\mu_X + \nu\nu_X) - K_X(\kappa_X^2 - 1) + K_Y(\kappa_X\kappa_Y - \kappa_Z) + K_Z(\kappa_X\kappa_Z - \kappa_Y)\}(X) = K\{\kappa - (s)\} \quad \ldots\ldots(5).$$

Similar equations are found for (Y) and (Z). This equation either means that (X) denotes orthogonal planes meeting in a line parallel to s, or the coefficient of (X) vanishes and $(s) = \kappa = 1$. The former is not of interest for the same reason as in cases two and three, and the latter relates to a different problem as in case one. We conclude, quite irrespective of the number and order of the reflections which take place, that no prism of the type sought with only four optical surfaces is possible.

Prisms with five surfaces

We have then to consider prisms in which an additional direction for the reflecting faces is present. As before we will denote the entrance and exit direction by o and the directions of the first and last reflecting surfaces by 1 and 2, the new direction being 3. When neither o surface is used for reflection we are led as before to equation (4) where X, Y, Z now denote the directions 23, 31, 12. The foregoing analysis of the cases which may arise shows that the only groups which may be of use are those in which the coefficient of (Z) vanishes but not those of (X) and (Y). Surface 3 must be parallel to the axis s.

We see at once that the only representative with four reflections, (1312)*, fails.

* Interchange of 1 and 2 and reversal of the order is essentially no change of type and all such duplicates are disregarded.

For brevity in examining systems in which the number of reflections is increased it is convenient to write x, y, z for $2\kappa_X, 2\kappa_Y, 2\kappa_Z$ respectively. The κ condition thus becomes $\alpha x + \beta y + \gamma z + 2\delta = 0$. The six possible types are analysed in Table I.

Table I

Type	α	β	γ	δ	Conditions
121312 or $ZY'Z$	0	-1	$yz - x$	0	Fails
131212 or $Y'ZZ$	z	1	yz	$-y - xz$	Fails
131312 or $Y'Y'Z$	y	0	$y^2 - 1$	$-xy$	Fails
131232 or $Y'ZX'$	x	$-y$	$-xy$	$1 - x^2 - y^2$	Fails
132312 or $Y'XZ$	yz	y	$xy - z$	$1 - xyz$	Fails
132132 or $Y'Z'X'$	$-x$	$-y$	$z - xy$	$x^2 + y^2 - 1$	$z = xy$, $x^2 + y^2 = 2$

Thus the only case in which a right angle may occur between the surfaces 1, 2, and 3 is in the second arrangement, when surfaces 1 and 3 must meet at right angles. Obviously this implies the failure of the system, for the reflected light will not meet surface 3 in a finite distance since this surface is also to be parallel to the s axis. The fifth system gives $l_1 = yz\Delta$, $l_2 = y\Delta$, where Δ is the determinant constructed from the direction cosines of surfaces 1, 2, and 3. Now $\kappa_{0212} = zl_2 - l_1$, and therefore vanishes, showing that the incidence for the fifth reflection is grazing. On the other hand the last system does not violate any of the conditions we are considering. Without loss of generality we may put $n_3 = 1$, and obtain for the direction cosines of the sides the values

$$(0)\ (1, 0, 0);\quad (1)\ \left(\sin\phi, \frac{\cos\phi}{\sqrt{2}}, \frac{\cos\phi}{\sqrt{2}}\right);\quad (2)\ \left(\cos\phi, -\frac{\sin\phi}{\sqrt{2}}, \frac{\sin\phi}{\sqrt{2}}\right);\quad (3)\ (0, 0, 1).$$

Now
$$\kappa_{01} = l_1,\ \kappa_{0131} = yl_1 - l_3,\ \kappa_{013231} = (xy - z)\,l_1 + l_2 - xl_3,$$
$$\kappa_{023132} = l_1 + (xy - z)\,l_2 - yl_3,\ \kappa_{0232} = xl_2 - l_3,\ \kappa_{02} = l_2,$$

and the cosines of the angles of incidence are $\sin\phi$, $\dfrac{\sin 2\phi}{\sqrt{2}}$ and $\cos\phi$ at each reflection at surfaces 1, 3, and 2 respectively.

When the number of reflections is increased to eight the number of types is found to be twenty-two. There is no difficulty in finding the coefficients $\alpha, \beta, \gamma, \delta$ for each of them. These quantities are given in Table II.

By inspecting these coefficients it is seen that the eight combinations marked with an asterisk are not suitable for inverting prisms. Of the remaining fourteen x, y, and z can be given values which will enable γ to be zero while α, β, and the cosine of the incident angle differ from zero in five only. The remaining nine fail through the incidence being grazing at the surfaces marked with a circumflex.

We may now turn to the other group of prisms in which one or both of the 0 surfaces are used for reflection. Since there are now four directions to be included the method we have hitherto followed will not in general yield an expression for (s) consisting of the sum of matrices each involving only two elements. The advantages of these simpler matrices in giving digestible information about the system are, however, so great that it appears advantageous to divide the system

into two parts each of which contains not more than three elements. In cases that are likely to arise in practice one part will not involve more than two elements, as will be seen in the systems investigated below.

Table II

Type	α	β	γ	δ
$13232312^{\bullet}$	x^2-1	0	$x^2y-xz-y$	$x(2-x^2)$
$\hat{1}323\hat{1}312$	xy	1	$y(xy-z)$	$-x^2y$
$\hat{1}323\hat{1}232$	x^2-1	$xy-z$	$x(xy-z)$	$x(2-x^2-y^2)+yz$
$\hat{1}323\hat{1}\hat{2}12$	xz	x	$1+xyz-z^2$	$z-xy-x^2z$
$\hat{1}323\hat{2}132$	$1-x^2$	$-xy$	$x(z-xy)$	$x(x^2+y^2-2)$
13132312	$xy-z$	-1	xy^2-yz-x	$y+xz-x^2y$
$\hat{1}313\hat{1}312^{\bullet}$	y^2-1	0	$y(y^2-2)$	$x(1-y^2)$
$\hat{1}313\hat{1}\hat{2}32^{\bullet}$	xy	y^2-1	$x(y^2-1)$	$y(2-x^2-y^2)$
$\hat{1}313\hat{1}\hat{2}12$	yz	y	$z(y^2-1)$	$1-xyz-y^2$
13132132	$z-xy$	$1-y^2$	$x+yz-xy^2$	$y(x^2+y^2-2)-xz$
$\hat{1}3\hat{1}\hat{2}1\hat{3}12^{\bullet}$	$yz-x$	0	$y(yz-x)$	$x^2-xyz-1$
$\hat{1}3\hat{1}\hat{2}1232$	xz	yz	$xyz-1$	$z(1-x^2-y^2)$
$\hat{1}3\hat{1}\hat{2}1\hat{2}12$	z^2-1	z	$y(z^2-1)$	$x-xz^2-yz$
$\hat{1}3\hat{2}1\hat{2}312^{\bullet}$	$xz-y$	0	$1+xyz-y^2-z^2$	$z+xy-x^2z$
$\hat{1}3\hat{2}1\hat{2}132$	$-xz$	$-yz$	$z^2-xyz-1$	$z(x^2+y^2-1)$
$1212\hat{3}2312^{\bullet}$	0	$-x$	$1+xyz-x^2-z^2$	z
$\hat{1}2\hat{1}3\hat{1}312^{\bullet}$	0	$-y$	$y^2z-xy-z$	1
$\hat{1}2\hat{1}3\hat{1}232$	y	$yz-x$	$1+xyz-x^2$	$-y^2z$
$\hat{1}2\hat{1}3\hat{1}\hat{2}12^{\bullet}$	1	0	$z(yz-x)$	$-yz$
$1212\hat{3}2132$	$-y$	$x-yz$	$x^2+z^2-xyz-1$	$z(y^2-1)$
$12\hat{3}2\hat{1}\hat{2}12$	$-z$	-1	xz^2-yz-x	y
$123\hat{1}\hat{2}312$	y	x	$x^2+y^2+z^2-xyz-1$	$-xy-z$

Let us assume that it has been found possible to divide the matrix into two parts each consisting of an even number of surfaces. The matrix equation may then be written $(s)=(pq)$, and this may be converted into either $(p)=(sq')$ or $(q)=(p's)$. As (s) is supposed to be given the problem is reduced in form to that we have already discussed.

The simplest possible form of this type is (1032), the only arrangement possible with four reflections. Taking $(s)=(0\ 1\ 0\ 0)$ we at once obtain

$$(\kappa_{10}\ \lambda_{10}\ \mu_{10}\ \nu_{10})=(-\lambda_{23}\ \kappa_{23}\ \nu_{23}\ -\mu_{23}),$$

conditions which readily lend themselves to a satisfactory arrangement. The surfaces are easily arranged to form a convex prism. One of the most convenient forms is obtained by letting surfaces 2 and 3 form a roof. The surface 1 will preferably make an angle greater than 45° with surface 0, so that the light tends to cross the thickness of the prism thrice. Two forms are shown in Figs. 2 and 3 (p. 91).

In interpreting the conditions to be found for prisms of this class it should be remembered that the surface 0 is normal to the axis s, which is in reality the axis of x, so that the λ of the combinations 10, 20, 30 vanishes. Thus in the example given above it is essential for surfaces 2 and 3 to meet in a right angle. Also the

intersection of no two reflecting surfaces should be parallel to the s axis, as both extreme surfaces 1 and 2 must make non-zero acute angles with the incident and emergent light. If then the conditions take the form

$$(sab) = \alpha(s_1) + \beta(s_2) + \gamma(s_3) + \delta$$

the three coefficients α, β, γ may not all be finite nor may they all be zero. There will in any particular case be no difficulty in deciding which coefficients must vanish.

When there are six reflections twenty distinct arrangements have to be considered. We will retain (X), (Y), (Z) with their old meanings and denote (10), (20), (30) by (U), (V), (W) respectively and define u, v, w analogously to x, y, z. The analysis of the twenty cases is given in Table III.

Table III

Type	s combination	(U)	(V)	(W)	(X)	(Y)	(Z)	Constant term
123012	(s)	0	0	1	0	0	$2\kappa_{1230}$	$-w$
131012	$(s21)$	$-y$	0	-1	0	0	0	0
	$(31s)$	0	1	0	0	0	u	$-v$
103012	$(s21)$	w	0	0	0	1	0	$-y$
130312	$(s21)$	-1	0	0	0	$-w$	0	wy
101032	$(s23)$	u	0	0	0	0	0	-1
121032	$(s23)$	z	-1	0	0	0	0	0
131032	$(s23)$	y	0	-1	0	0	0	0
120132	$(s23)$	$-z$	1	0	0	0	u	0
130132	$(s23)$	$-y$	0	1	0	$-u$	0	uy
130102	$(s20)$	$-y$	0	1	0	$-u$	0	uy
	$(31s)$	0	$-u$	0	0	0	-1	uv
132102	$(s20)$	0	0	0	-1	$-z$	$-y$	$x + yz$
	$(31s)$	1	$-z$	0	0	0	$-v$	vz
103102	$(s20)$	y	0	-1	0	u	0	$w - uy$
123102	$(s20)$	0	0	0	1	z	y	$-yz$
131202	$(s20)$	0	0	0	1	0	y	$-x$
	$(31s)$	-1	0	0	0	0	v	0
123202	$(s20)$	0	0	0	0	1	x	$-y$
	$(21s)$	0	0	-1	$-v$	0	0	vx
101302	$(s20)$	0	0	1	0	$-u$	0	$uy - w$
130302	$(s20)$	-1	0	0	0	$-w$	0	wy
	$(31s)$	0	$-w$	0	1	0	0	$vw - x$
121302	$(s20)$	0	0	0	-1	$-z$	0	yz
131302	$(s20)$	0	0	0	0	$-y$	0	$y^2 - 1$
132302	$(s20)$	0	0	0	0	$-x$	-1	xy
	$(31s)$	0	$-x$	1	v	0	0	0

A glance at the table shows the advantage of dividing the system in two ways where this can be done. Take for example the system 132102, which is separated into such a number of components that at least one coefficient must be made equal to zero. According to the first resolution we are left in doubt whether we should select y or z to be made zero; the second form shows that we must have either v or z equal to zero, and as both u and v must be finite we can say at once that we must let z vanish. In fact the only case in which the term to disappear is in doubt

at the first glance is 123102. This doubt is resolved as soon as the solution is considered in detail, as will shortly appear.

The precise meaning of the table and the way in which it may be used can be seen from the following examples. For the first system the table indicates that

$$(s) = (W) + 2\kappa_{1230}(Z) - w,$$

or
$$(0\ 1\ 0\ 0) = (l_3\ 0\ n_3 - m_3) + 2\kappa_{1230}(\kappa_{12}\ \lambda_{12}\ \mu_{12}\ \nu_{12}) - 2l_3,$$

that is
$$\frac{\kappa_{12}}{l_3} = \frac{\lambda_{12}}{1} = \frac{\mu_{12}}{-n_3} = \frac{\nu_{12}}{m_3} = \frac{1}{2\kappa_{1230}} = \sqrt{\left\{\frac{\kappa_{12}^2 + \lambda_{12}^2 + \mu_{12}^2 + \nu_{12}^2}{l_3^2 + 1 + n_3^2 + m_3^2}\right\}} = \frac{1}{\sqrt{2}},$$

conditions which there should be no difficulty in realising in practice. Take next the last arrangement. From the second resolution it is clear that we must have $x = 0$, and substituting this value in the first form we obtain

$$(0\ l_2 - m_2 - n_2) = -(\kappa_{12}\ \lambda_{12}\ \mu_{12}\ \nu_{12}),$$

so that
$$l_1 = l_2\kappa_{12} + m_2\nu_{12} - n_2\mu_{12} = 0,$$

showing that the solution fails. Now consider the arrangement 123102 already mentioned. The relation indicated by the table may be written in the slightly simplified form

$$(0\ l_2 - m_2 - n_2) = (\kappa_{23}\ \lambda_{23}\ \mu_{23}\ \nu_{23}) + 2\kappa_{12}(0\ \lambda_{31}\ \mu_{31}\ \nu_{31}) + 2\kappa_{31}(0\ \lambda_{12}\ \mu_{12}\ \nu_{12}),$$

showing that surfaces 2 and 3 are perpendicular to one another. Multiplying throughout by $(0\ l_2\ m_2\ n_2)$ gives $2l_2^2 - 1 = 2\kappa_{12}\Delta$, where Δ is the determinant formed from the direction cosines of surfaces 1, 2, and 3. Thus $\kappa_{12} = 0$ involves $l_2 = 1/\sqrt{2}$, a value unacceptable in a system in which the last two reflections are 02. We must therefore suppose that $\kappa_{31} = 0$: then multiplying by $(0\ l_3\ m_3\ n_3)$ gives $2l_2l_3 = \kappa_{23} = 0$, and since $l_2 = 0$ cannot be admitted we take $l_3 = 0$. This condition in conjunction with $\kappa_{23} = \kappa_{31} = 0$ must be rejected, and the arrangement thus fails to yield a useful solution.

On general principles we should anticipate that the failure of a solution with this number of reflecting directions would be exceptional. The three surfaces usually represented on the right side of the equation have in all six degrees of freedom, and four conditions have to be satisfied. When failure occurs it will usually be due, as in the two examples just discussed, to the failure of the light to reach some surface at which reflection takes place within a finite distance, or through two directions coinciding. The latter event is always obvious, and a criterion has been given in the previous paper for the former. A detailed analysis of the systems having eight reflections at surfaces in four different directions cannot be presented in this paper on account of the space that would be required. The number of distinct types to be considered is 240. Systems in which several reflections take place at few surfaces tend to have a smaller field of view than is attainable when the number of reflections is less owing to the combined effect of the conditions to be satisfied at each reflection.

Boundary conditions

The boundary conditions for each surface of the systems which satisfy the tests here considered must be examined before we can say definitely that the proposed construction is possible with only five surfaces. Their importance can be seen from a simple illustration.

When there is only one surface in each of the different directions 1, 2, 3, they all meet in a point. If there is no reflection at one of the 0 surfaces this point is a self-conjugate point of the initial and final spaces irrespective of the number of reflections which take place. In our problem any solution with a self-conjugate point necessarily fails, for it implies zero loss of time through deviation of the path while the light traverses the system. We may therefore conclude that the systems of Tables I and II which have survived the tests already applied to them involve at least six optical surfaces instead of the five we have assumed. The additional surface or surfaces will of course lie in one or more of the directions 1, 2, 3.

APPENDIX

To illustrate in detail methods of completing a solution we will consider two simple systems in which the conclusions may be verified from first principles either by drawing or by simple geometry. Take first the prism with two reflections illustrated in Fig. 1, the reflecting surfaces of which, as we have already noticed, form

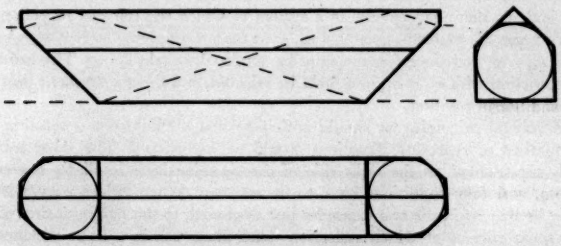

Fig. 1. Prism with two reflections

a right-angled roof. Let the edge of the roof be the axis of x, so that the equations of the reflecting surfaces may be represented by $y + z = 0$ and $y - z = 0$. Let the incident and emergent beams be confined within the cylinder

$$(y - b)^2 + z^2 = a^2.$$

The roof surfaces must not cut this cylinder, but they will touch it in a prism of

minimum size. Substitution for z in this equation from the equations of the roof surfaces yields

$$(y-b)^2+y^2=a^2 \text{ or } (2y-b)^2=2a^2-b^2.$$

For the smallest prism the two values of y will be equal, or

$$y=\tfrac{1}{2}b=a/\sqrt{2}.$$

Let μ be the index of refraction of the prism, and ϕ and ϕ' the angles of incidence and refraction as the light enters the prism. The ray along the axis of the incident cylinder then makes an angle $\phi-\phi'$ with the roof edge within the prism. From symmetry it emerges along the axis of the emergent cylinder and is reflected at the central point of the roof edge. If then 2ρ is the length of the path in glass and $2l$ is the length of the prism, we easily find

$$b=\rho\sin(\phi-\phi'),$$

$$l=a\tan\phi+\rho\cos(\phi-\phi')=a\{\tan\phi+\sqrt{2}\cot(\phi-\phi')\},$$

$$\sin\phi=\mu\sin\phi'.$$

We can determine the minimum value of l for given values of a and μ by computing its value for a few selected values of ϕ and plotting the connexion between l and ϕ graphically. Alternatively we may differentiate the equation for l with respect to ϕ and solve the equation for $\sin\phi$ resulting from equating this differential coefficient to zero.

As a second example consider symmetrically constructed prisms of the type 1032, two forms of which are shown in Figs. 2 and 3. Let the origin be where the last three reflecting surfaces meet, and take $z=0$ as the plane of symmetry. Then if l_2, m_2, n_2 are the direction cosines of surface 2, those of surface 3 are $l_2, m_2, -n_2$, and since $\kappa_{23}=0$ we have $l_2^2+m_2^2-n_2^2=0$. We may therefore write

$$\frac{l_2}{\sin\theta}=\frac{m_2}{\cos\theta}=\frac{n_2}{1}=\frac{1}{\sqrt{2}}.$$

The conditions

$$(\kappa_{10}\ \lambda_{10}\ \mu_{10}\ \nu_{10})=(-\lambda_{23}\ \kappa_{23}\ \nu_{23}-\mu_{23})$$
$$=(l_1\circ n_1-m_1),$$

at once give $l_1=\cos\theta$, $m_1=\sin\theta$, $n_1=0$. The equations to the reflecting surfaces are therefore

surface 1 $\quad x\cos\theta+y\sin\theta=p,$

,, 0 $\quad x=0,$

,, 3 $\quad x\sin\theta+y\cos\theta-z=0,$

,, 2 $\quad x\sin\theta+y\cos\theta+z=0.$

It will be observed that the angles made by BC and AE with AB or CD, that is, the angles made with 0 by 1 and the intersection of 2 and 3, are complementary. To avoid any need to silver the surfaces the angle ABC should be the greater.

Since only surface 1 fails to pass through the origin we can write without calculation

$$(R) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 2p\cos\theta & 2p\sin\theta & 0 & 1 \end{pmatrix}.$$

Equation (4) of the previous paper then gives

$$x' = x + 2p\cos\theta,$$

$$y' = -y + 2p\sin\theta,$$

$$z' = -z$$

as the equations of transformation from the initial to the final space. Thus the line $y = p\sin\theta$, $z = 0$ is its own image, but no point on it is self-conjugate. It is evidently the axis of the screw transformation represented by (R), and the axial displacement in the half turn is $2p\cos\theta$. If the distance between the entrance and exit surfaces is h, the length of the path in the prism will be $h + 2p\cos\theta$. Recollecting that it is preferable to make $\theta = ABC > 45°$, $x = 0$ will be the exit surface and $x = -h$ the entrance surface.

As in the previous example we will assume that the light to be transmitted lies within a circular cylinder of radius a. The equation of the incident cylinder will be $(y-b)^2 + z^2 = a^2$ and that of the emergent cylinder $(y'-c)^2 + z'^2 = a^2$ where $b + c = 2p\sin\theta$. When the prism is of minimum size the intersections of surface 1 with the entrance and exit surfaces will touch the incident cylinder. These intersections are $x = -h$, $y = p\operatorname{cosec}\theta + h\cot\theta$ and $x = 0$, $y = p\operatorname{cosec}\theta$ respectively. We must therefore have

$$p = (b-a)\sin\theta, \quad h = 2a\tan\theta.$$

Similarly, the intersections of surfaces 2 and 3 with the exit surface $x' = 0$ must touch the emergent cylinder. From the equations of these surfaces $z' = \pm y'\cos\theta$, and the equation

$$(y'-c)^2 + (y'\cos\theta)^2 = a^2$$

should have equal roots. That is to say,

$$(1 + \cos^2\theta)(c^2 - a^2) = c^2,$$

or

$$c = a(1 + \sec^2\theta)^{\frac{1}{2}}.$$

If we note that the width of the prism is $2a$ and the length of the exit face $b - c$, we have all the relations needed to determine the dimensions of the prism for any values of a and θ.

The prism illustrated in Fig. 2 is a limiting case of the prisms which it is con-

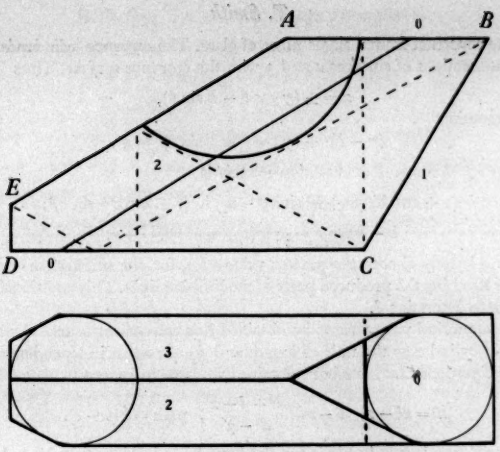

Fig. 2. Prism of type 1032 with five optical surfaces

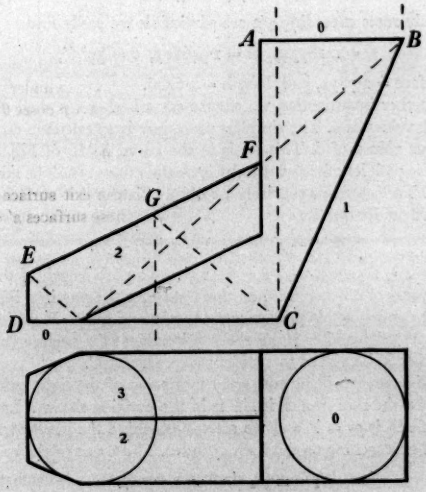

Fig. 3. Composite prism of type 1032

venient to construct from a single piece of glass. The entrance cylinder is touched by the intersections of surfaces 2 and 3 with the entrance surface. Thus

$$z = \pm (y \cos\theta - h \sin\theta),$$

and the equation

$$(y - b)^2 + (y \cos\theta - h \sin\theta)^2 = a^2$$

must have equal roots. The condition for this is

$$(1 + \cos^2\theta)(b^2 + h^2 \sin^2\theta - a^2) = (b + h \cos\theta \sin\theta)^2,$$

or

$$b - h \tan\theta = \pm a (1 + \sec^2\theta)^{\frac{1}{2}} = \pm c.$$

We must evidently choose the greater value of b, for the smaller corresponds to a cylinder touching the produced parts of the limiting lines. This additional relation enables us to determine θ.

When numerical values are to be obtained it is convenient to take the diameter of the light cylinder as the unit of length and $d = 2c$ as an independent variable. From the equations that have been obtained we have in general

$$h^2 = d^2 - 2, \quad b - c = \frac{d^2 + d - 2}{d^2 - 3}, \quad \text{path} = h(b - c).$$

In the special case shown in Fig. 2 we also have $b - c = h^2$, so that d is to be found from the equation

$$(d^2 - 2)(d^2 - 4) = d.$$

Noting that only roots exceeding $\sqrt{2}$ are admissible we easily find

$$d = 2{\cdot}187549, \quad h = 1{\cdot}946805, \quad \theta = 59^\circ\, 6',$$

length of exit face = 2·7854, path length = 4·6486.

Without further investigation we cannot say whether we shall obtain shorter paths for other values of θ. The simplest procedure is to calculate the value of the path for various values of d. This leads to the curve KML of Fig. 4, where the ordinates are the path lengths and the abscissae the values of θ. The minimum value is found at M. To locate this precisely we differentiate $h(b - c)$ with respect to d, and are thus led to the quintic

$$d(d^2 - 2)(d^2 - 5) = 2(2d^2 - 3).$$

The solution is $d = 2{\cdot}59005$, with $\theta = 65^\circ\, 15'$ and a path length of 4·2705.

Since this value of θ is greater than that found for the previous form, it is clear that we cannot construct this prism from one piece of glass. It could be made of two pieces cemented together along the continuation of a surface such as AF in Fig. 3, which should be placed to avoid a sharp edge at C.

From the diagram it will be suspected that some of the light reflected near B will be cut off by the roof near F if AF is in the position shown. Let us suppose the equation of AF is $y_1 = Y$ and we are to determine the greatest value we can give Y without encroaching on the light cylinder. The traces of the roof on AF are

$$z_1 = \pm (x_1 \sin\theta + Y \cos\theta),$$

and the y and z coordinates of the image of these lines in surface 1 are

$$y = Y + 2 \sin\theta\,(p - x_2 \cos\theta - Y \sin\theta),$$

$$z = \pm\,(x_1 \sin\theta + Y \cos\theta).$$

Substituting these values in the equation of the incident cylinder gives

$$\{Y - b + 2 \sin\theta\,(p - x_1 \cos\theta - Y \sin\theta)\}^2 + \{x_1 \sin\theta + Y \cos\theta\}^2 = a^2.$$

This equation is to give equal values of x_1, or

$$Y\,(1 + 2\cos 2\theta) + 2p \sin\theta - b = \pm\, a\,(3 + 2\cos 2\theta)^{\frac{1}{2}}.$$

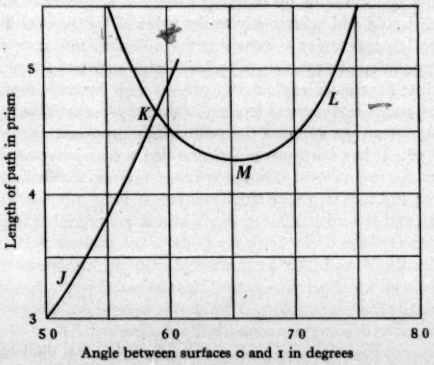

Fig. 4. Relation between length of path and angle of prism of type 1032

Substituting the values already found for p, b, θ shows that Y may not lie between 1·7733 and 3·9438. Thus the greatest value Y may have is 1·7733. This involves a small eclipse of the exit cylinder, since $y = c + a = 1{\cdot}7947$ at the point G. It would appear preferable to replace the plane AF by one lying approximately in the direction AG, and the new limiting conditions may be investigated in the same way as the illustrations already given.

Fig. 4 shows an alternative method of finding the system of Fig. 2. The curve KML shows the path length when the exit cylinder is assumed to be limited by the roof surfaces, and the curve JK is similarly obtained when the entrance cylinder is so limited. The point K where the two curves cross one another determines the system.

The glass in the neighbourhood of A bounded by a cylinder, indicated by an arc of a circle in Fig. 2, may be removed as is customary in the larger prism found in optical instruments.

DISCUSSION

Capt. T. Y. Baker: Mr Smith's two papers on the design of reflecting prisms deal, but in an entirely different manner, with the same problem as two papers I read last year. He has treated the question from an analytical and algebraic standpoint and has evolved a method that is full of mathematical elegance. But I am by no means sure that it will appeal to the optical designer whose mathematical equipment is small and who will undoubtedly have great difficulty in following Mr Smith's papers. In this respect I think Mr Smith has failed to make the most of his opportunities by omitting to give a fully worked-out numerical example which would add enormously to the value of the papers. I trust that he will take an early opportunity of repairing the omission and show *in extenso* the numerical computation of a problem of optical design such as I am constantly being confronted with; he should explain the process step by step showing how the numbers of the computation are arrived at, so that anyone unable to appreciate the underlying theory can yet appraise the value of his proposals as a practical computation method. It has been my experience that the majority of cases of prism design concern the necessity of taking a focussed beam and twisting it round into a new position; and that in a very large number of cases it is the direction of the emergent beam and the orientation of the field that are of higher importance than the actual distance of the ocular from the body of the instrument.

I should like to see worked out a numerical example of this character and I am sure it would be well worth printing in the *Transactions*. I would be only too pleased to provide the data for the problem. I make this suggestion, also, because I think Mr Smith has underestimated the numerical difficulties of the boundary conditions when trying to compress the axial glass path to the minimum dimensions. He has stated in general terms the method which must be followed, of expressing, in some form or other, an equation to the bounding surface of the beam and putting down a condition that the line of intersection of two successive reflecting planes must just touch the surface of the beam and not cut into it. But stating the method in general terms and working out numerical cases are two very different things and I should much like to see a numerical case worked out in detail.

One thing Mr Smith seems to have overlooked. Fields are not always circular and the bounding surface is extremely frequently *not* a cone. The divided field of a rangefinder and the elongated field that is most suitable for reading an extended scale such as occurs in the more elaborate methods of reading a theodolite are cases in point. The design in many of these cases has been carried out for the full circular field that will contain the area required and the unwanted positions blanked out, but where economy in space and glass path are required such a proceeding does not conduce to the best results. If Mr Smith can show that the methods he advocates will enable the designer to deal with cases of this character he will have made a most valuable contribution to the art of designing optical instruments.

The classification of systems of reflectors that is proposed does not seem to me

quite a practical one, although it is satisfactory from the purely mathematical standpoint. The designer is generally faced with the problem of getting the light from one position to another and wants to know the least number of reflecting surfaces that will meet the case. There is, as Mr Smith points out, a clear-cut division into two classes: an odd number of reflections is necessary when a right-handed object is to be converted into a left-handed image; an even number when the image is also right-handed. In the latter case, as I showed in my earlier paper, the most general case can be dealt with by four reflections, but the solution, so far from being unique, is multiply infinite. The general case of the odd number is similar. Three reflections are necessary and there is a quadruply infinite set of solutions.

Those two classifications are general. There is for the odd-numbered case the exceptional one of the self-conjugate locus being a plane where one single reflection will do what is required. There is for the even-numbered case a similar exception of the self-conjugate locus being a straight line, a case for which two plane reflections are adequate. Beyond that point it seems to me unnecessary to go. Everything can be solved, in theory, by three reflections for the right-to-left case, by four reflections for the right-to-right one. In stating that fact I do not suggest that all cases can in practice be dealt with in this manner, but where it is necessary to go to five or to six the two additional surfaces are imposed by some external conditions such as the necessity of having a prism that does not foul some other part of the instrument.

In conclusion I must once more call attention to Mr Smith's weakness, as it seems to me, of treating a problem, which in the majority of cases is purely directional, by means of rectangular cartesian analysis. It involves him in the expression of directions in terms of parameters which convey to the mind no immediate idea of those directions relatively to the instrument and I strongly suspect that the whole of the analysis of binocular prism systems could be performed diagrammatically upon the surface of a sphere.

Mr Smith: Capt. Baker has raised many points in his written remarks and still more verbally. Several of them in effect are equivalent to asking why I have introduced complexities into a simple subject when I might have adopted one of a number of simpler methods, preferably the one recommended by Capt. Baker. The answer is that my method is obviously both simpler and better than any of the alternatives suggested. In some respects the difference is not unlike that between two methods of getting a number of pigs to market. The old way was to place each pig in the charge of a man who could impart that twist to the tail which, if it did not always produce forward, at least involved no retrograde motion. The modern farmer prefers to place all his pigs in a lorry driven by a man of somewhat higher training than the tail-twisters. When he has once appreciated the advantages of employing the lorry and its skilled driver he will not be deflected from his course by neighbours who object that many of his hands do not understand how the engine works and cannot drive. In case of emergency some of them could be taught to drive, though they might be incapable of understanding the mechanism put in their

charge. If this is a fair simile when we are comparing graphic and matrix methods, we may compare the spherical trigonometric method with the conveyance of a pig by a succession of carriers. No carrier goes direct between farm and market, and the route is thus roundabout and changes are unavoidable. There is also a risk that the carrier may be going in the other direction that day so that the pig may be taken to the wrong market after all, or may get lost before the carrier arrives on his return journey.

Coming now to some points of detail, the second paper is written on the supposition that the directional problem is the more important, and all essential details have been given for about 50 classes of prisms. Anyone who follows these through will need no numerical illustrations. The ease with which an exhaustive examination may be made is a notable feature of this method.

I am unable to agree with Capt. Baker's view regarding the difficulty offered by boundary conditions. The conical envelope mentioned was only intended as an illustration. As this part of the method has been considered obscure I have added an appendix dealing in full detail with some special examples. I am prepared to show a solution by my methods of such a problem as Capt. Baker has in mind, but unless solutions by other methods are offered for comparison workers interested in these problems will have no standard by which the merits and demerits of the method can be judged.

The classification I have suggested is based on the view that the properties of the transformation are of greater significance than the number of surfaces involved.

Finally there is Capt. Baker's parting shot at the weakness of my methods of dealing with purely directional problems. I can assure Capt. Baker that I hold no brief for any one method, and am ready to change as soon as he can demonstrate that I shall gain more than I lose. I believe the most rapid progress is made when we allow the problem itself to dictate the tools that should be employed. The theory of axially symmetrical lenses has been mentioned. Have any notable results in that subject been attained by the use of either polar or cylindrical coordinates? If not can we say why? The problem we have been discussing in these two papers will help us to supply an answer to the general question. We have found that the direction cosines of an emergent ray are given by linear functions with constant coefficients of the incident direction cosines, the simplest mathematical relations we know. To find corresponding formulae for variables corresponding to latitude and longitude we have only to substitute $\cos\theta\cos\phi$, $\cos\theta\sin\phi$, $\sin\theta$ for l, m, n respectively in our formulae. We at once see that it is only in very exceptional cases that we obtain a linear relation in these variables. Both theory and calculation are very much simpler with the rectangular coordinates. The apparent simplicity of the angular system is in fact misleading in many ways. We can, to my mind, derive a much more exact mental picture of direction with the rectilinear than with the angular system. Moreover we must not forget that when we write $\cos\theta$, $\sin\theta$, $\tan\theta$, we are employing, not one variable, but three. True when one is given the others can be found, though ambiguities of sign remain: but as far as the labour involved in numerical work goes, the significant fact is that they are different quantities involving special calculation no less than if they had been functions of a new variable.

REVIEWS

Gesammelte Abhandlungen von Ernst Abbe, Band IV, erste Hälfte, Arbeiten zum Glaswerk zwischen 1882 *und* 1885. With Introduction by Professor Moritz von Rohr. Pp. xiv + 146; 3 Figs. and photograph of Abbe. (Jena: Gustav Fischer, 1928.) Price, unbound 7 RM., bound 9 RM.

This collection of papers deals with Abbe's contributions to the foundation of optical glass manufacture in Jena at the glass-works of Schott and Gen. In England the researches of Faraday and, later in the nineteenth century, of Harcourt and Stokes into glass manufacture on a laboratory scale had been of considerable importance, but these men had been handicapped through lack of co-operation with an outstanding technical optician. The combination of Abbe, with his clear foresight of the trend of technical optics, of Schott, who through his family connection had early directed his chemical skill to the problems of glass melting, and of the technical assistance of the Zeiss works was a powerful one. Abbe states that since Fraunhofer's time the manufacture of glass had separated itself from Optics and that there was in 1882 a lack of scientific knowledge in the glass industry. Optical glass was practically a monopoly of two or three firms in France and England and these firms specified their glasses by specific gravity and, so far as they were concerned, Fraunhofer might never have existed. The problems specified for solution by Abbe and Schott were: (*a*) what chemical elements occurring in what combinations give amorphous colourless glass melts, (*b*) what is the contribution of these elements and compounds to refractive index and dispersion, (*c*) how can glass of a specified refractive index be produced? Abbe pointed out in what directions the glasses available at that time were wanting. He desired to reduce the secondary chromatic aberration. It had always up to this time been assumed that the refractive indices of flint glass must be greater than those of crown. Early melts by Schott showed that this was not necessarily so. Crown glasses ranging from 1·48 to 1·65 were produced instead of the then listed range of 1·50 to 1·53. Abbe also set for solution the problem of the physical attributes, especially the coefficient of expansion, with special reference to thermometer glass and glow lamps. The manufacture of such special glassware was later to form a very large part of the Schott undertaking. The problem of state help can be followed in detail from the letters and acts which Prof. von Rohr has collected from the Kultusministerium in Berlin. Both Abbe and Schott were of the opinion that, though work on a small scale could be carried out with the funds at their disposal, yet factory scale manufacture was impossible without state aid. Bamberg (p. 34) states that Schott threatened that, if state aid were not forthcoming, he might be reluctantly compelled to offer his researches to Feil of Paris or Chance of Birmingham. The state aid actually granted was small: 25,000 Marks in 1884/5 and 35,000 Marks in 1885/6. The interesting letter of E. Busch shows how much the Rathenow optical industry of this time was dependent on foreign, especially French glass. Fortunately for the Rathenow industry the war against the French in 1871 was of short duration, but probably the military needs were a strong factor for the granting of even limited state help. The military problems of the great war similarly compelled us to make up the leeway. There is an excellent description of the expenses and lay-out of the original glass factory and a personal appreciation by Abbe of Schott. To the student of the history of optical glass this carefully edited volume is as much a necessity as Prof. von Rohr's researches on the earlier contributions of P. Guinand and Fraunhofer.

N. M. S. L.

An Atlas of Ophthalmoscopy. By Dr O. Haab. Translated and edited by W. B. Barker, F.B.O.A. Pp. 230; 87 Figs. (London: British Optical Association, 1928.) Price, bound in cloth, 25*s*., in calf, 31*s*. 6*d*.

This is an old book: old in age, old in use, and old in the way the subject matter is presented. In its original form it was a small atlas that could be slipped in a none too capacious pocket and be used as a vade-mecum by the junior student of ophthalmology. It has served this function to many ophthalmologists practising at the present. Now, after the lapse of well over a quarter of a century, the atlas has made a new appearance—in an entirely new garb and is intended specially for the use of opticians. The subject matter remains what it was; the only change the plates have undergone is that they loom larger in space: they are surrounded by a waste of blank paper. These plates which were very good in 1902, are not quite so good in 1928; the present editor does not seem to have seen some of the modern plates. In the introductory matter explaining the optical principles underlying the construction and use of the ophthalmoscope, space has been found for some remarks on the use of red-free light. This appears to be the only advance in ophthalmology to which the editor has responded. The British Optical Association is in many ways a most active body: its eagles are to be found in every street; but in the case of the publication of an atlas of the fundus oculi it seems to have held fast to the truth of Haab, rather than being bent on the more venturesome course of "proving all things."

A. H. L.

Colour Photography. By Capt. Owen Wheeler, F.R.P.S. Pp. x + 138; 2 Figs and 15 Plates. Supplement: *Colour Snapshots*. Pp. viii; 1 Fig. (London: Sir Isaac Pitman & Sons, Ltd.; Henry Greenwood & Co., Ltd., 1928.) Price 12*s*. 6*d*. net.

This book deals with colour photography methods from the standpoint of the amateur and is written in a chatty or popular manner. It gives detailed and comprehensive instructions for working all the more important processes and pays particular attention to three-colour Carbro, and it is to this method that several chapters are devoted.

There is little attempt in this book to deal with fundamental principles and in fact there is no indication as to which colours are likely to present difficulties in reproduction by subtractive methods, or which are easy to reproduce. The reviewer has not discovered any attempt to deal with the degradation of greens for example which is so constant a cause of trouble. The difficulties of all the methods are dealt with most adequately from the point of view of manipulative technique.

The style in which the book is written rather favours the occurrence of small slips, of which there are several, but these are not of a nature to put an intelligent photographer on the wrong track except in one most unfortunate instance. On page 41 very precise instructions are given for diluting concentrated sulphuric acid and the instructions are that the water should be "poured on top of the acid"! It is to be hoped that no unfortunate person unfamiliar with chemical operations will follow this instruction, as it might easily lead to serious bodily injuries. It is a pity that the volume contains such errors, but these are for the most part slight, and any intelligent photographer with some knowledge of chemistry could derive much useful information on three-colour processes from this book.

A. J. BULL.

Penrose's Annual: The Process Year Book and Review of the Graphic Arts. Vol. xxxi. Edited by Wm. Gamble, F.R.P.S., F.O.S. Pp. xvi + 184. (London: Percy Lund, Humphries & Co., Ltd., 1929.) Price 8*s.* net.

The present volume contains the following articles: "The Editor's Review," "The Nonesuch Press," "Modern Printing in all Countries," "The Work of the First Edition Club," "Advertisement in the Near Future," "The New Colour Photography and the Art of the Painter," "Progress in Colour Photography," "Where would we be without the Camera?," "Printers can Learn from Advertising Agents," "New Silver Baths for Old Ones," "Three Simple Ways of Getting a Woodcut Effect," "A New Screen," "The Use of the 'Hilite' Lens," "On the Use of Line Blocks," "Standardising Three-colour Inks," "A System to Aid Colour Interpretation," "A Simplified Method of Screen Negative Making," "Selling Engraving Work," "Organisation amongst Master Printers and Allied Groups in the United States," "American Artists tend to Study Graphic Technique," "Commercial Typography Five Years Hence," "Type and Paper, Old and New: Contrasts and Comparisons," "Transparent Paper," "Modern English Book Illustration," "A Note on Bookplates," "The Bindings of Books," "Present Day Printing in France," "The Gregynog Press," "Fancy Paper," "Gummed Paper—Its Varieties and their Uses," "Photogravure from Thin Copper Plates," "The Field for Pantone," "Dry Lithography," "The Posters of the Empire Marketing Board," "On Lino-Cutting for Schools," "Two-Revolution or Wharfedale? A Comparison," "Making Photo-Litho Pay," "A New Way in Collotype," "Elka: A New Process," "The Step and Repeat Projector," and "The Editor's Note Book." The volume contains numerous illustrations representative of process work during the year 1928, in addition to plates illustrating the articles.

Bureau of Standards Journal of Research. (Washington: United States Government Printing Office.) Subscription $2.75 per year (United States, Canada, and Mexico), $3.50 (foreign countries).

This is a new journal which the Bureau of Standards is issuing monthly; the first number appeared in July, 1928. It supersedes the two series of research publications previously issued by the Bureau as "Scientific Papers" and "Technologic Papers."

ABSTRACTS FROM PATENT SPECIFICATIONS

279900*. British Thomson-Houston Co., Ltd. OPTICAL SLITS.

A narrow slit for use in optical sound-recording apparatus is produced by obtaining a negative of a dark line on a light ground, or by making a contact print of a fine wire. The negative is varnished, and protected from the heat of the lamp by a coating of aluminium foil.

279907*. Siemens und Halske Akt.-Ges. LIGHT VALVES.

In a light valve for use in television or talking-picture apparatus, a transparent body placed between crossed polariser and analyser is subjected to direct mechanical stresses in accordance with the signals, etc. to be transmitted or recorded.

279930. Allom, G. REFLECTORS.

A reflector for interior lighting consists of an unsymmetrical part-spherical part-parabolic surface, the parts being of unequal area, and so arranged as to project two beams in directions which cross each other. The surface may be continuous or formed from strips, and the glass may be waved or crinkled for purposes of diffusion.

280351. Marks, E. C. R. LIGHT DIFFUSERS.

The source of light is surrounded by a series of concentric diffusing reflectors so arranged that all light is incident upon them at more than 45 degrees to the normal. Direct light may be intercepted by a silvered reflector which throws back the light upon the diffusers.

280392. Warmisham, A. and another. LENSES.

A divergent member consisting of a cemented doublet meniscus having its convex outer surface directed towards the longer conjugate is placed between two convergent elements, of which one is a cemented doublet and the other a singlet.

280552*. Zeiss, C. COLORIMETERS.

The instrument measures the colouring matter added to a coloured or turbid solvent. A comparison is made of the light transmitted through two adjustable systems one of which contains the solvent and a standard solution of the colouring matter, the other including the solvent and the solution under examination, both diluted to half strength.

R. J. T.

* Specification published before acceptance.